Lecture 17:

Frequency Response of Amplifiers

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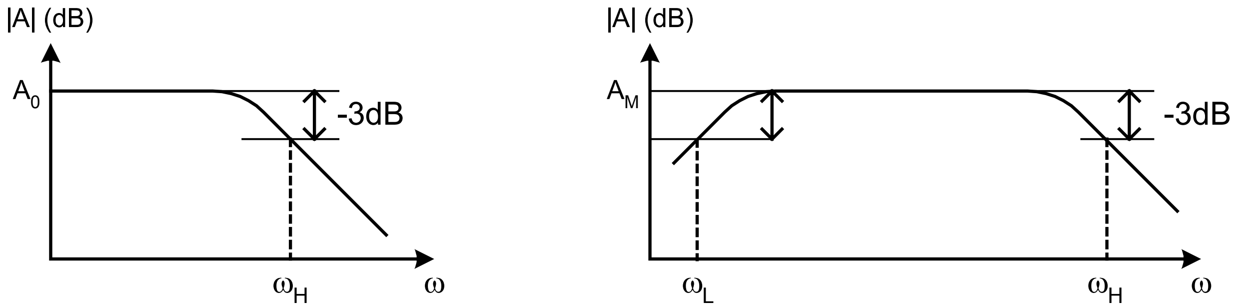
Overview

- **Reading**
  - S&S: Chapter 7
    - Skim sections since mostly described using BJT circuits. Lecture notes focus on MOS circuits.
- **Supplemental Reading**
  - Razavi, *Design of Analog CMOS Integrated Circuits*: Chapter 6
- **Background**
  - So far, our treatment of small-signal analysis of amplifiers has been for low frequencies where internal capacitances do not affect operation. However, we did see that internal capacitances do exist and we derived the f_T of transistors. Moreover, we spent some time looking at amplifiers modeled with a single pole. Now, we will see how these capacitances affect the frequency response of amplifiers.

    To fully understand and model the frequency response of amplifiers, we utilize Bode plots again. We will use a technique called open-circuit time constants (OCTs) to approximate frequency response calculations in the presence of several capacitors and and Miller's theorem to deal with bridging capacitors.

Amplifier Transfer Function



- Voltage-gain frequency response of amplifiers seen so far take one of two forms
  - Direct-Coupled (DC) amplifiers exhibit low-pass characteristics – flat gain from DC to ω_H
  - Capacitively coupled amplifiers exhibit band-pass characteristics – attenuation at low frequency due to impedance from coupling capacitances increasing for low frequencies
- We will focus on the high-frequency portion of the response (ω_H)
  - Gain drops due to effects of internal capacitances of the device
- Bandwidth is the frequency range over which gain is flat
  - $BW = \omega_H$ or $\omega_H - \omega_L \approx \omega_H$ ($\omega_H >> \omega_L$)
- Gain-Bandwidth Product (GB) – Amplifier figure of merit
  - $GB \equiv A_M \omega_H$
    where A_M is the midband gain
  - We will see later that it is possible to trade off gain for bandwidth

Gain Function A(s)

- We can represent the frequency dependence of gain with the following expression:

$$A(s) = A_M F_L(s) F_H(s)$$

  – Where $F_L(s)$ and $F_H(s)$ are the functions that account for the frequency dependence of gain on frequency at the lower and upper frequency ranges
  – We can solve for A_M by assuming that large coupling capacitors are short circuits and internal device capacitances are open circuits (what we have done so far for low-frequency small-signal analysis)

High-Frequency Response

- We can express function FH(s) with the general form:

$$F_H(s) = \frac{(1+s/\omega_{Z1})(1+s/\omega_{Z2})\cdots(1+s/\omega_{ZnH})}{(1+s/\omega_{P1})(1+s/\omega_{P2})\cdots(1+s/\omega_{PnH})}$$

- Where ω_P and ω_Z represent the frequencies of high-frequency poles and zeros
- The zeros are usually at infinity or sufficiently high frequency such that the numerator → 1 and assuming there is one dominant pole (other poles at much higher frequencies), we can approximate the function as…

$$F_H(s) = \frac{1}{(1+s/\omega_{P1})}$$

$$\omega_H \cong \omega_{P1}$$

  – This simplifies the determination of the BW or ω_H
- If a dominant pole does not exist, the upper 3-dB frequency ω_H can be found from a plot of $|F_H(j\omega)|$. Alternatively, we can approximate with following formula (see S&S p593 for derivation).

$$\omega_H = 1 \Big/ \sqrt{\frac{1}{\omega_{P1}^2} + \frac{1}{\omega_{P2}^2} + \cdots - \frac{2}{\omega_{Z1}^2} - \frac{2}{\omega_{Z2}^2}\cdots}$$

  – Note: if ω_{P1} is a dominant pole, then reduces to $\omega_H = \omega_{P1}$

Open-Circuit Time Constant Method

- It may be difficult to find the poles and zeros of the system (which is usually the case). We can find approximate values of ω_H using the following method.

  – We can multiply out factors and represent $F_H(s)$ in an alternative form:

  $$F_H(s) = \frac{1 + a_1 s + a_2 s^2 + \cdots + a_{nH} s^{nH}}{1 + b_1 s + b_2 s^2 + \cdots + b_{nH} s^{nH}}$$

  – Where *a* and *b* are coefficients related to the zero and pole frequencies

  – We can show that $$b_1 = \frac{1}{\omega_{P1}} + \frac{1}{\omega_{P2}} + \cdots + \frac{1}{\omega_{PnH}}$$

  and b_1 can be obtained by considering the various capacitances in the high-frequency equivalent circuit one at a time while reducing all other capacitors to zero (or open circuits); and calculating and summing the RC time constant due to the circuit associated with each capacitor.

  – This is called the open-circuit time constant method (OCT)

Calculating OCTs

The approach:

- For each capacitor:
  - set input signal to zero
  - replace all other capacitors with open circuits
  - find the effective resistance (R_{io}) seen by the capacitor C_i
- Sum the individual time constants (RCs or also called the open-circuit time constants)

$$b_1 = \sum_{i=1}^{nH} C_i R_{io}$$

- This method for determining b_1 is exact. The approximation comes from using this result to determine ω_H.

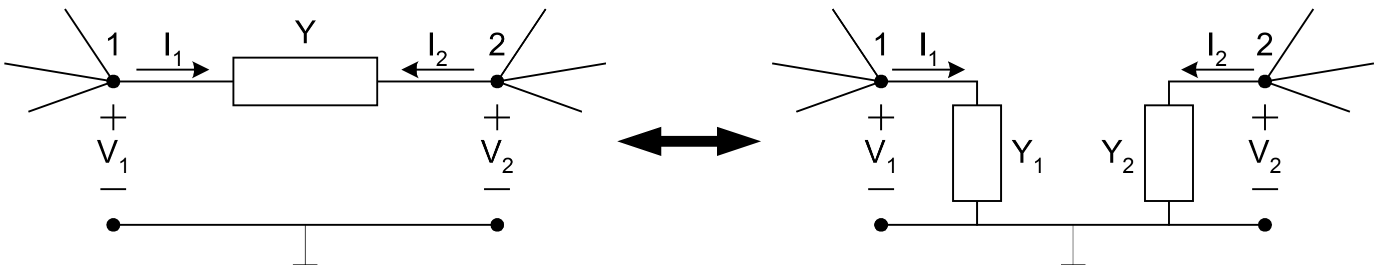
$$\omega_H \cong \frac{1}{\sum_{i=1}^{nH} C_i R_{io}}$$

  - This equation yields good results even if there is no single dominant pole but when all poles are real
- We will see an example of this method when we analyze the high-frequency response of different amplifier topologies

Miller's Theorem

- Before we begin analyzing the high-frequency response of amplifiers, there is an important phenomenon that we should first investigate called "Miller Effect"

  Consider the circuit network below on the right with two nodes, 1 and 2. An admittance Y (Y=1/Z) is connected between the two nodes and these nodes are also connected to other nodes in the network. Miller's theorem provides a way for replacing the "bridging" admittance Y with two admittances Y1 and Y2 between node 1 and gnd, and node 2 and gnd.



  – The relationship between V_2 and V_1 is given by $K=V_2/V_1$
  – To find Y_1 and Y_2

$$I_1 = Y(V_1 - V_2) = YV_1(1 - V_2/V_1)$$
$$I_1 = YV_1(1-K)$$
$$I_1 = Y_1V_1$$
$$Y_1 = Y(1-K)$$

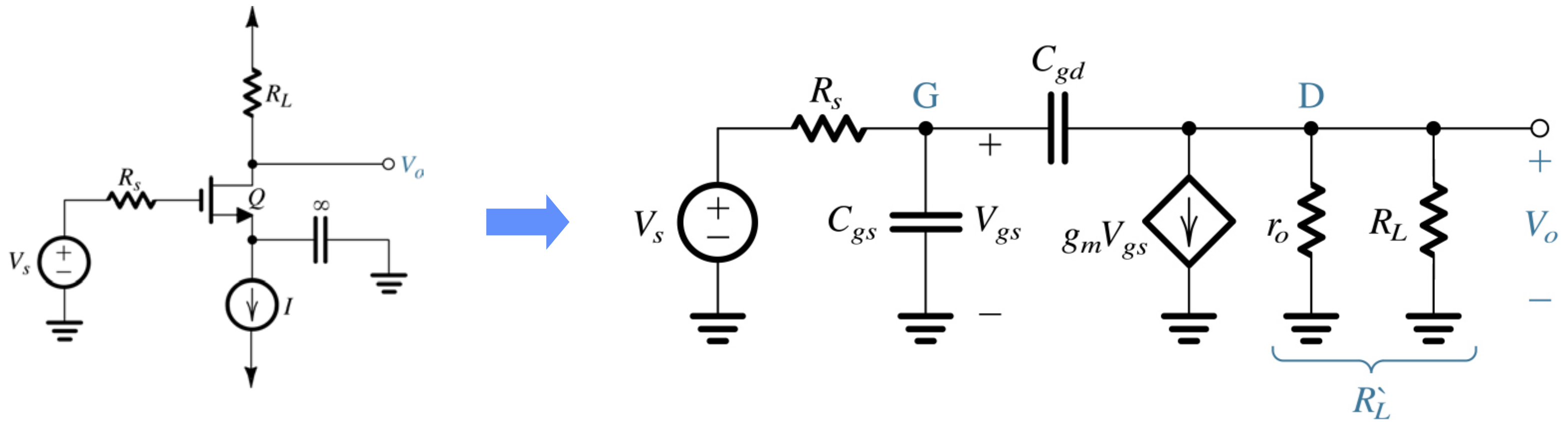
$$I_2 = Y(V_2 - V_1) = YV_2(1 - V_1/V_2)$$
$$I_2 = YV_2(1-1/K)$$
$$I_2 = Y_2V_2$$
$$Y_2 = Y(1-1/K)$$

Caveat:

The Miller equivalent circuit is valid only as long as the conditions that existed in the network when K was determined are not changed.

High-Frequency Response of CS Amp

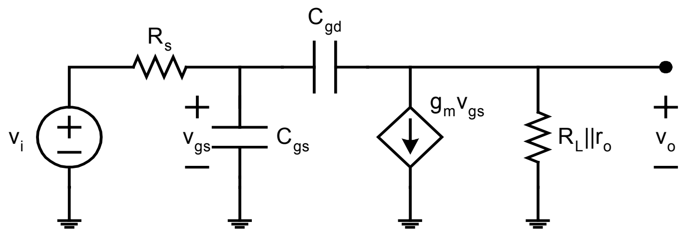
- Take the following circuit and investigate its high-frequency response
  - First, redraw using a high-frequency small-signal model for the nMOS



- There are two ways to find the upper 3-dB frequency ω_H
  - Use open-circuit time constant method
  - Use Miller's theorem
  - Brute force calculations to find v_{out}/v_{in}
- Let's investigate them all

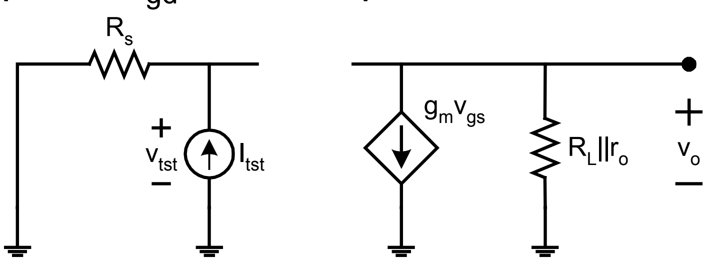
Using OCT on CS Amplifier

- Find the RC time constants associated with C_{gd} and C_{gs} in the following circuit



$$A_M = \frac{v_o}{v_i} = -g_m\left(R_L \parallel r_o\right)$$

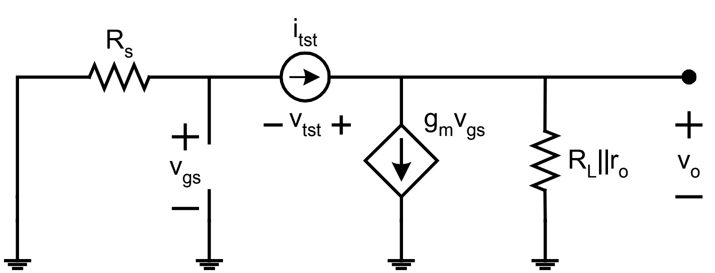
- Replace C_{gd} with an open-ckt and find the resistance seen by C_{gs}



$$R_{gs} = \frac{v_{tst}}{i_{tst}} = R_s$$

$$\tau_{gs} = R_{gs}C_{gs} = R_s C_{gs}$$

- Replace C_{gs} with an open-ckt and find the resistance seen by C_{gd}



$$i_{tst} = -v_{gs}/R_s \qquad i_{tst} = g_m v_{gs} + \frac{v_{gs} + v_{tst}}{R_L \parallel r_o}$$

$$R_{gd} = v_{tst}/i_{tst} = \left(R_L \parallel r_o\right) + g_m R_s\left(R_L \parallel r_o\right) + R_s$$

$$\tau_{gd} = R_{gd}C_{gd} = \left[\left(R_L \parallel r_o\right) + g_m R_s\left(R_L \parallel r_o\right) + R_s\right]C_{gd}$$

Using OCTs Cont'd

- Summing to two time constants yields ω_H

$$\omega_H \cong \frac{1}{\tau_{gs} + \tau_{gd}}$$

$$\omega_H \cong \frac{1}{R_s C_{gs} + \left[\left(R_L \parallel r_o \right) + g_m R_s \left(R_L \parallel r_o \right) + R_s \right] C_{gd}}$$

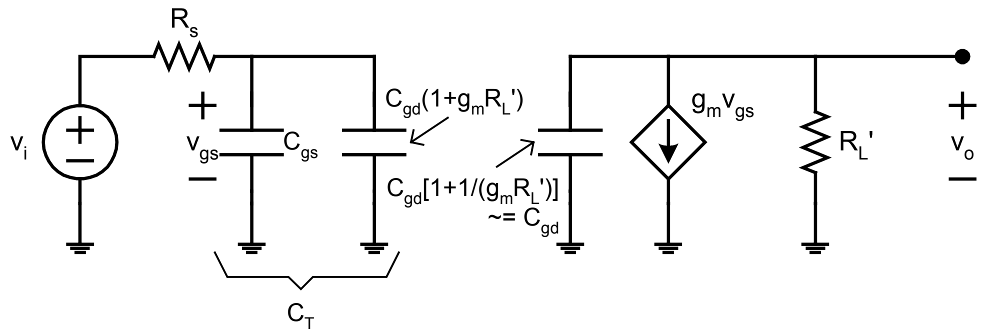
  - From the above equation, it is not difficult to imagine that C_{gd} has a more significant effect on reducing BW
  - The resulting frequency dependence of gain is…

$$A(s) = \frac{A_M}{s/\omega_H + 1}$$

- Let's compare this result with what we get using Miller's theorem

Using Miller's Theorem on CS Amplifier

- Redraw the high-frequency small-signal model using Miller's theorem



  – Assuming a dominant pole introduced by C_{gd} in parallel with C_{gs}

$$\omega_H \cong \frac{1}{\left[C_{gs} + C_{gd}\left(1 + g_m R_L'\right)\right] R_s} = \frac{1}{C_T R_s}$$

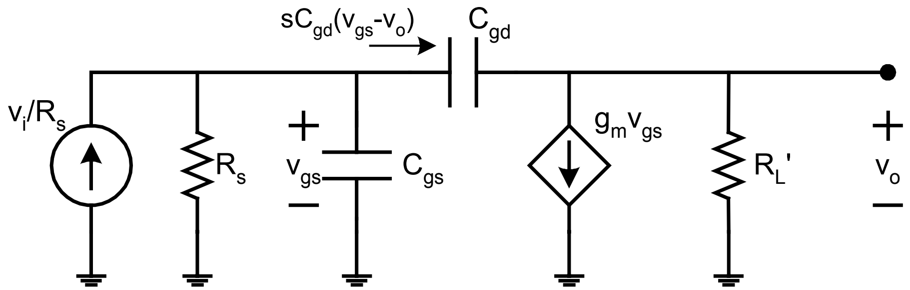
  – Miller multiplication of C_{gd} results in a large input capacitance

- Notice that this approximation for ω_H is close to the approximation found using OCT assuming that $R_s C_{gd}(1+g_m R_L')$ dominates

- Let's verify our assumptions by deriving the exact high-frequency transfer function of the CS amplifier

High-Frequency Response of CS Amplifier

- Replace the input source and series resistance with a Norton equivalent



$$\frac{v_i(s)}{R_s} = \frac{v_{gs}}{R_s} + sC_{gs}v_{gs} + sC_{gd}\left(v_{gs} - v_o\right)$$

$$sC_{gd}\left(v_{gs} - v_o\right) = g_m v_{gs} + \frac{v_o(s)}{R_L'}$$

$$\Rightarrow \frac{v_o(s)}{v_i(s)} = -A_0 \frac{1 - \dfrac{s}{g_m / C_{gd}}}{1 + s\left[R_s C_{gs} + R_s C_{gd}\left(1 + g_m R_L'\right) + C_{gd} R_L'\right] + s^2 C_{gs} C_{gd} R_s R_L'}$$

  - The exact solution gives a zero (at a high frequency) and two poles
  - Notice that the *s* term is the same as the solution using the OCT method

- Unfortunately, the denominator is too complicated to extract any useful info… So, assuming the two poles are widely separated (greater than an order of magnitude), we can rewrite the expression for the denominator as…

HF Response of CS Amplifier

- Rewrite the denominator as:

$$D(s) = (1 + s/\omega_{P1})(1 + s/\omega_{P2}) = 1 + s(1/\omega_{P1} + 1/\omega_{P2}) + s^2/\omega_{P1}\omega_{P2}$$

$$D(s) \cong 1 + s/\omega_{P1} + s^2/\omega_{P1}\omega_{P2}$$

  - And from the solution on the previous slide we can write…

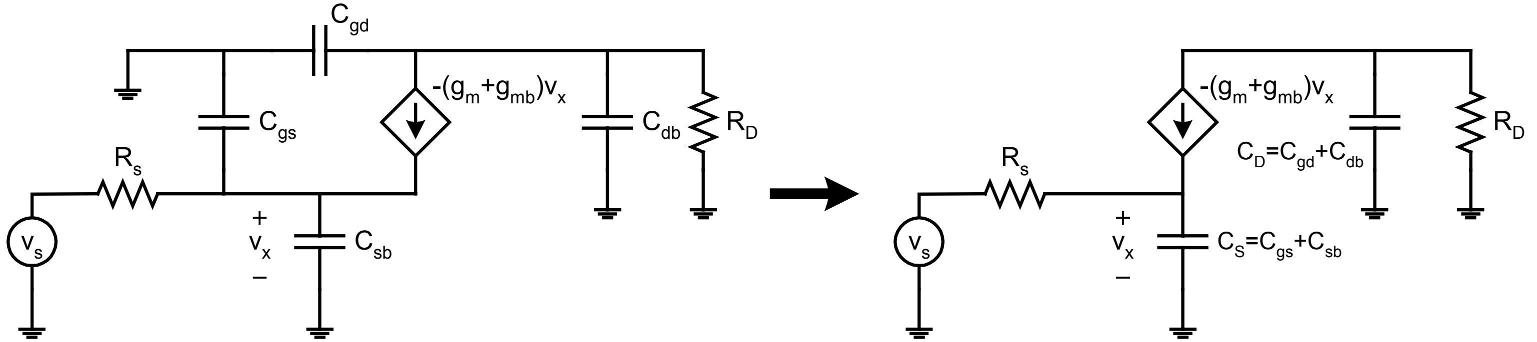
$$\omega_{P1} = \frac{1}{R_s C_{gs} + R_s C_{gd}(1 + g_m R_L') + R_s C_{gd}\, R_L'/R_s}$$

$$\omega_{P2} = \frac{C_{gs} + C_{gd}(1 + g_m R_L') + C_{gd}\, R_L'/R_s}{C_{gs} C_{gd} R_L'} \cong \frac{g_m}{C_{gs}}$$

- So the second pole is usually at a much higher frequency and we can assume a dominant pole
- Using either Miller's theorem or OCTs enables a way to quickly find approximations of the amplifier's high-frequency response

Frequency Response of CG Amplifier

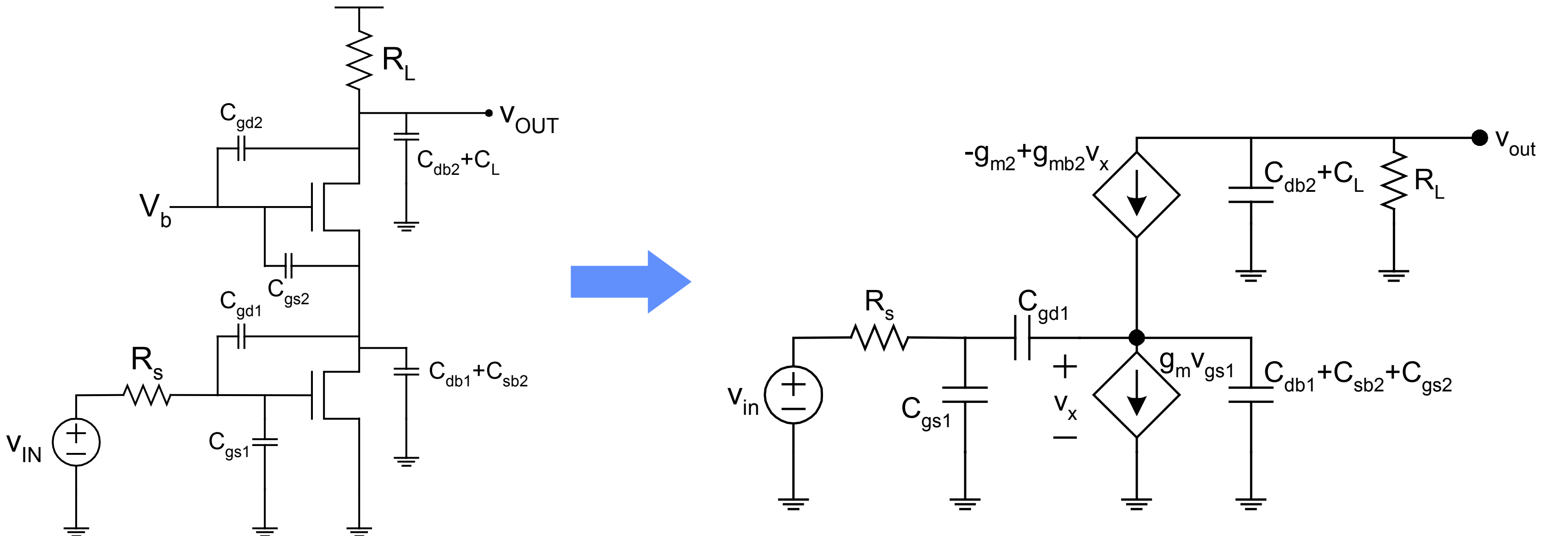
- One way to avoid the frequency limitations of Miller multiplication of C_{gd} is to utilize a CG amplifier configuration



- Using OCT method, we find two time constants
  - At the input (source node) $\tau_S = \dfrac{C_S}{1/R_s + g_m + g_{mb}}$
  - At the output (drain node) $\tau_D = C_D R_D$
- The output usually drives additional load capacitance such that the output pole is dominant
- The frequency response of CG amplifiers is when combined with a CS stage to build a cascode circuit

Cascode Stage

- Cascoding enables high bandwidth by suppressing Miller multiplication of C_{gd}. Let's investigate how with the following high-frequency model of a cascode stage.



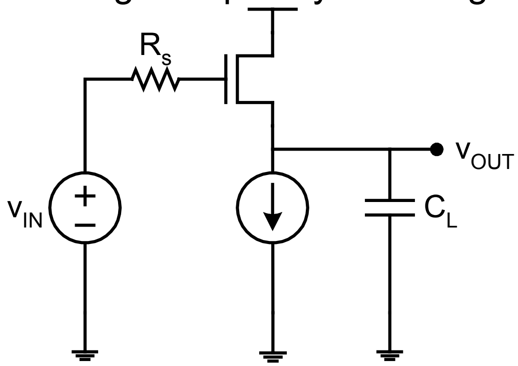
  - Use OCT method to find the time constants associated with each capacitor. The time constant associated with C_{gd1} is…

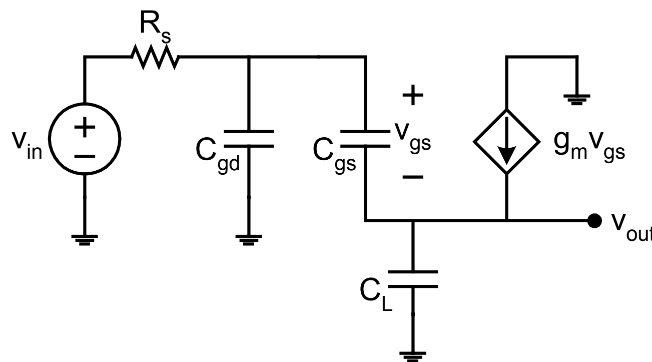
$$\tau_{gd1} = \frac{1 + g_{m1}R_s + (g_{m2} + g_{mb2})R_s}{g_{m2} + g_{mb2}} C_{gd1} \cong \left(1 + \frac{g_{m1}}{g_{m2} + g_{mb2}}\right) R_s C_{gd1}$$

NOTICE:
Miller multiplication is ~2

Frequency Response of Source Followers

- Start with a high-frequency small-signal model of the source follower circuit





  – Directly solving for v_{out}/v_{in} yields:

$$v_{gs}sC_{gs} + g_m v_{gs} = v_{out}sC_L \rightarrow v_{gs} = \frac{sC_L}{g_m + sC_{gs}} v_{out}$$

$$v_{in} = R_s\left[v_{in}sC_{gs} + \left(v_{in} + v_{gs}\right)sC_{gd}\right] + v_{gs} + v_{out}$$

$$\frac{v_{out}}{v_{in}}(s) = \frac{g_m + sC_{gs}}{R_s\left(C_{gs}C_L + C_{gs}C_{gd} + C_{gd}C_L\right)s^2 + \left(g_m R_s C_{gd} + C_L + C_{gs}\right)s + g_m}$$

$$\omega_{P1} \cong \frac{g_m}{g_m R_s C_{gd} + C_L + C_{gs}}$$

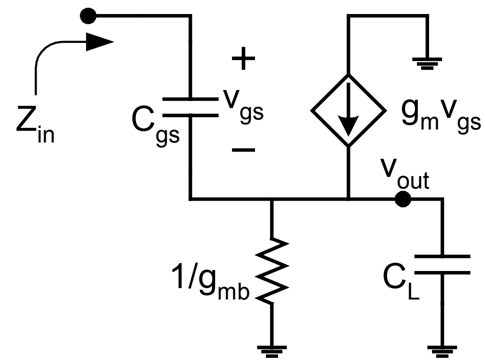
    - The zero is due to C_{gs} that directly couples the signal from the input to the output

  – If poles are far apart, then the *s* term represents the dominant pole

More on Source Follower

- Other important aspects of a source follower are its input and output impedances (since they are often used as buffers)
- Let's calculate the input impedance using the high-freq small-signal models

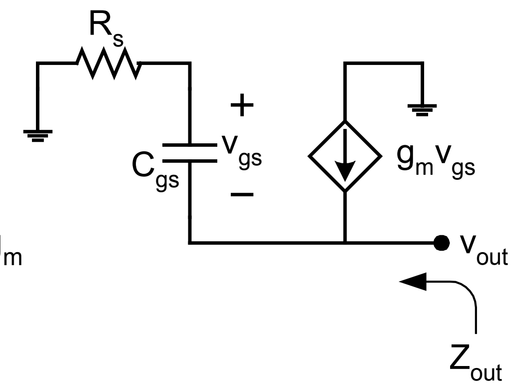
$$Z_{in} = \frac{1}{sC_{gs}} + \left(1 + \frac{g_m}{sC_{gs}}\right)\frac{1}{g_{mb} + sC_L}$$

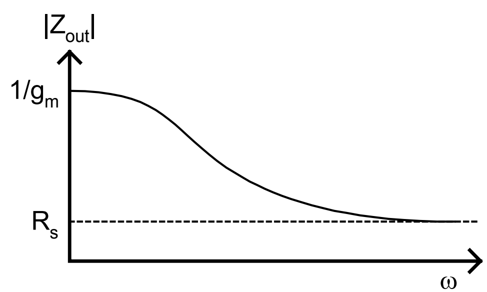


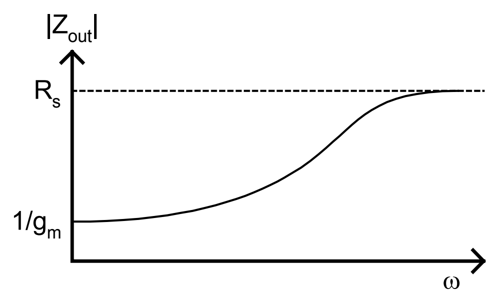
- Now calculate the output impedance (ignoring g_{mb} for simplicity)

$$Z_{out} = \frac{R_s sC_{gs} + 1}{g_m + sC_{gs}}$$

  - At low frequency, $Z_{out} \approx 1/g_m$
  - At high frequency, $Z_{out} \approx R_s$
  - Shape of the response depends on the relative size of R_s and $1/g_m$



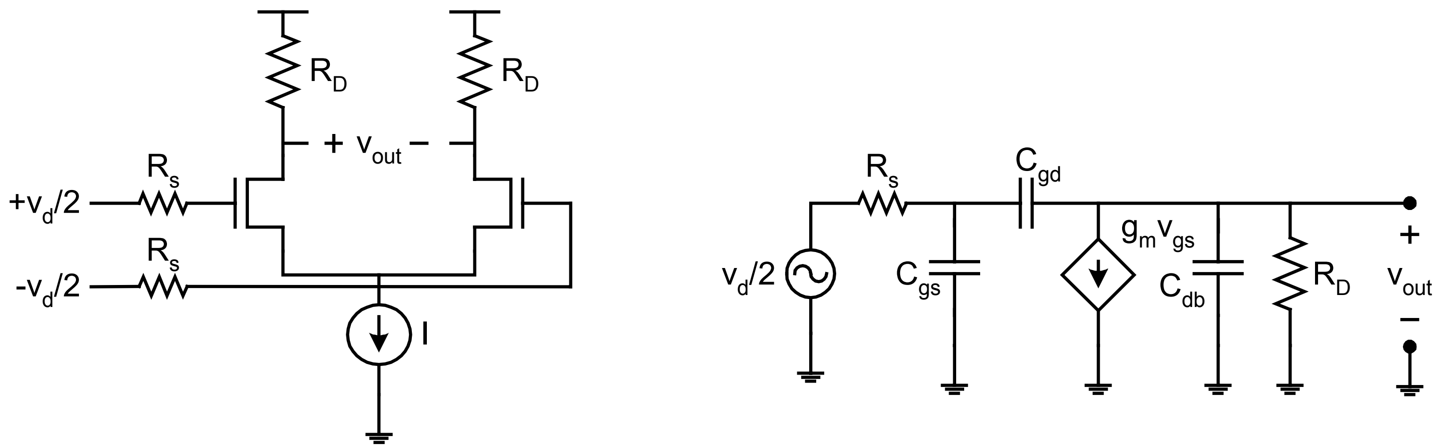




Z_{out} can look inductive or capacitive depending on R_s and $1/g_m$

Differential Pair

- We have seen that a symmetric differential amplifier can be analyzed with a differential half circuit. This still holds true for high-frequency small-signal analysis.



  – The response is identical to that of a common-source stage

High-Frequency CMRR

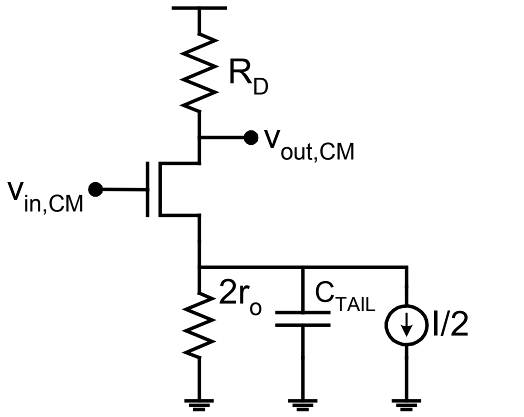
- The CMRR of a differential pair degrades at high frequency due to a number of factors. The most important is the increase in CM gain with frequency due to capacitance on the tail node.
- Use the common-mode equivalent half circuit to understand how CM gain increases with frequency
  - Draw the small-signal equivalent model and see the effect of C_{TAIL} on the v_{out}/v_{in} transfer function

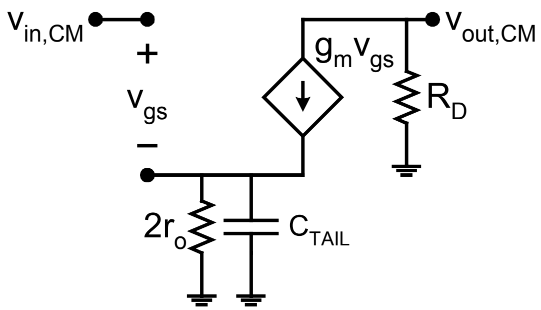
$$v_{out} = g_m v_{gs} R_D \text{ and } v_{gs} = v_{in} - v_x \text{ and } v_x = g_m v_{gs}\left(2r_o || 2/sC_{TAIL}\right)$$

$$v_{gs} = \frac{v_{in}}{1 + g_m\left(2r_o || 2/sC_{TAIL}\right)} \Rightarrow \frac{v_{out}}{v_{in}} = \frac{g_m R_D}{1 + g_m\left(2r_o || 2/sC_{TAIL}\right)}$$

$$\frac{v_{out}}{v_{in}}(s) = \frac{g_m R_D\left(1 + s r_o C_{TAIL}\right)}{1 + 2g_m r_o + s r_o C_{TAIL}}$$

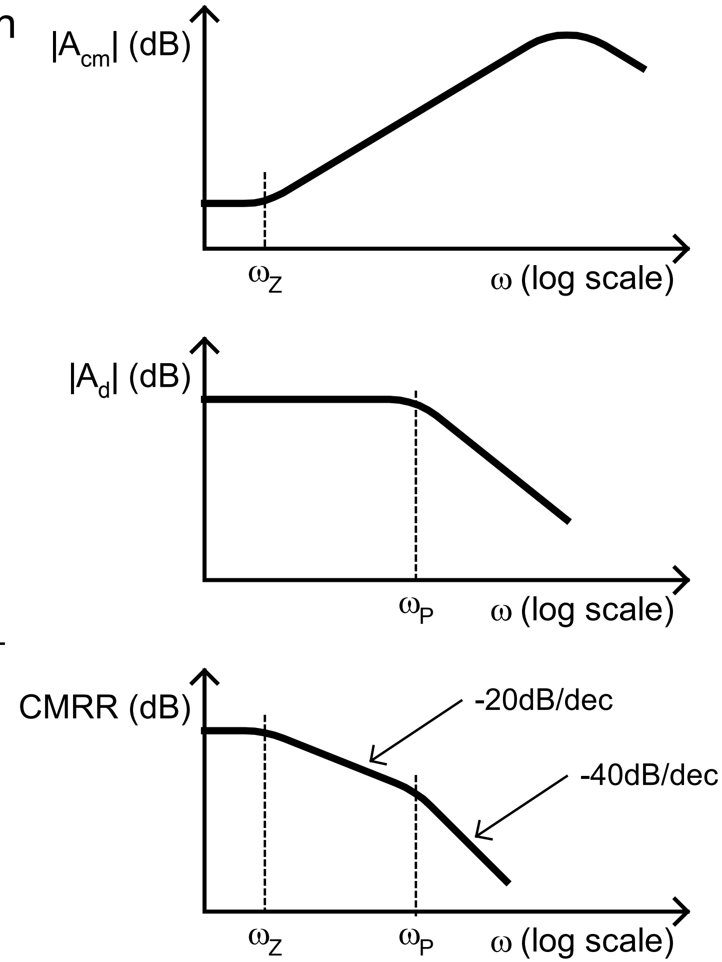
  - Zero at $\omega_Z = 1/r_oC_{TAIL}$ (since r_o is big, ω_Z occurs at a low frequency)
  - There are additional poles at higher frequencies due to C_{TAIL} and other internal capacitances (that we have ignored)
- The zero causes the CM gain to increase with frequency until the higher frequency poles kick in → CMRR degrades due to the zero





HF CMRR plots

- The impact of the zero in the CM gain on CMRR can be illustrated as shown
  - Remember CMRR = A_d/A_{cm}



- There is a trade off between CMRR and voltage headroom
  - Wider current source devices enable lower v_{ds}
  - Wider current source device means larger C_{TAIL}

Next Time

- Reading
  - S&S Chapter 8
- Supplemental Reading
  - Razavi: Chapter 8

- What to look forward to…
  - Negative feedback for amplifiers was invented in 1927 by Harold Black to stabilize the gain and correct the distortion of amplifiers used in long-distance telephone networks. Negative feedback (as well as positive feedback) is widely used in analog circuits today. In fact, we used negative feedback when we constructed op amps with gain set using resistors. Throughout the next lecture, we will investigate the general theory of feedback and look at four basic feedback topologies. We will also learn how to understand and analyze the stability of amplifiers.