Lecture 2:

Review of Basic Circuit Analysis (in time domain)

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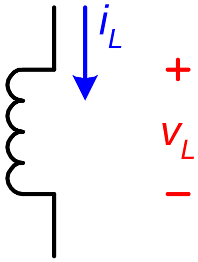
Overview

- **Reading**
- **Supplemental Reading**
  - Nilsson: Chapters 6-8

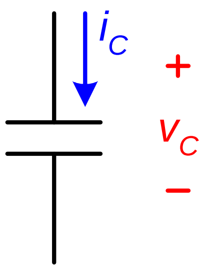
    (Most textbooks on Electric Circuits have this material)
- **Background**
  - In this lecture, we will continue to review basic circuit analysis and focus on circuits that contain reactive elements inductors and capacitors. Inductors and capacitors are a little more tricky than simple resistors b/c their current/voltage relationship also depends on time. For now, we will rely on differential equations to describe their transient behavior. In the next lecture, we will see how to use the Laplace transform to analyze circuits in the s-domain, which allows us to analyze circuit characteristics w.r.t. frequency.

L and C

- The current and voltage relationship of an inductor and capacitor are governed by the following equations:



$$v_L = L\frac{di_L}{dt} \qquad i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L \cdot dt + i_L(t_0)$$



$$v_C = C\frac{dv_C}{dt} \qquad v_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C \cdot dt + v_C(t_0)$$

- Some intuitive properties:
  - current through an inductor cannot change instantaneously, but voltage across one can
  - voltage across a capacitor cannot change instantaneously, but current through it can

Power and Energy in L and C

- Remember, $p = vi$ *and* $p = dw/dt$ (power is the rate of change of energy)
  - notice, I used *w* to represent energy as opposed to *E* to avoid confusion with electric fields
- Inductors store magnetic field energy

$$p_L = L i_L \frac{di_L}{dt} \Rightarrow p_L = \frac{dw}{dt} = L i_L \frac{di_L}{dt} \Rightarrow dw_L = L i_L di_L$$

  - Integrate on both sides

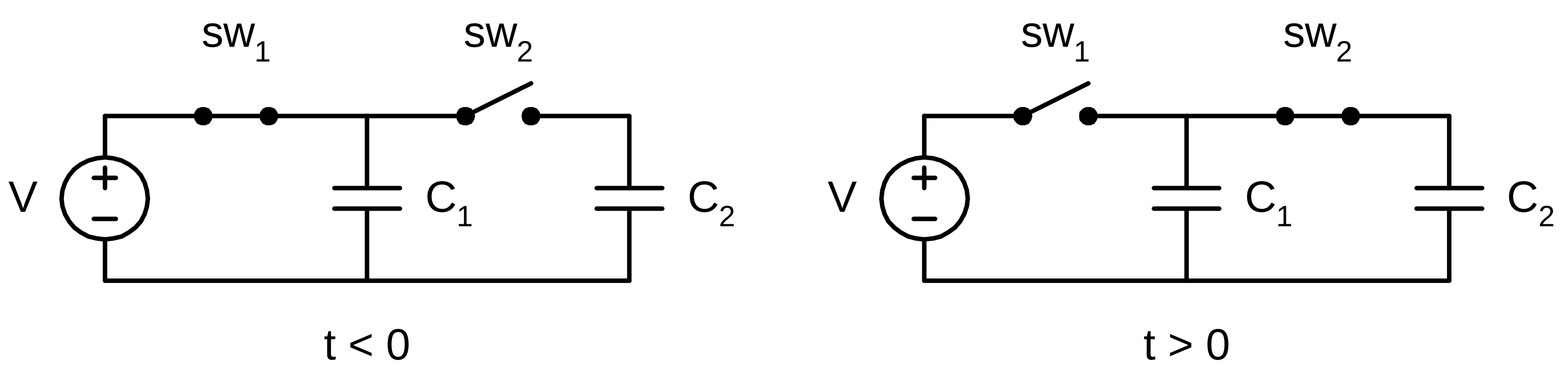
$$\int_0^w dx = L\int_0^i y \cdot dy \Rightarrow w_L = \frac{1}{2} L i_L^{\ 2}$$

- Capacitors store electric field energy

$$w_C = \frac{1}{2} C v_C^{\ 2}$$

Energy vs. Charge

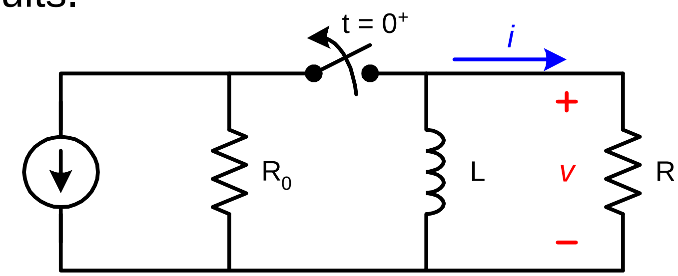
- A classic E&M problem…
  - Q = C*V and we just saw that w = ½ CV^2
  - Let's assume we have two capacitors. At t<0, C1 is charged by a voltage source. At t=0^+, the switch 1 (sw_1) opens and switch 2 (sw_2) closes to add an additional capacitor C_2 in parallel with C_1. Calculate the redistribution of charge and the redistribution of energy at t>0.

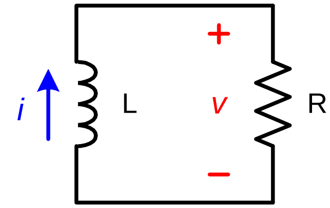


  - Conservation of energy vs. conversation of charge?

Natural Response of RL Circuit

- We can use the concepts we've learned so far to solve for the natural response of RL circuits.

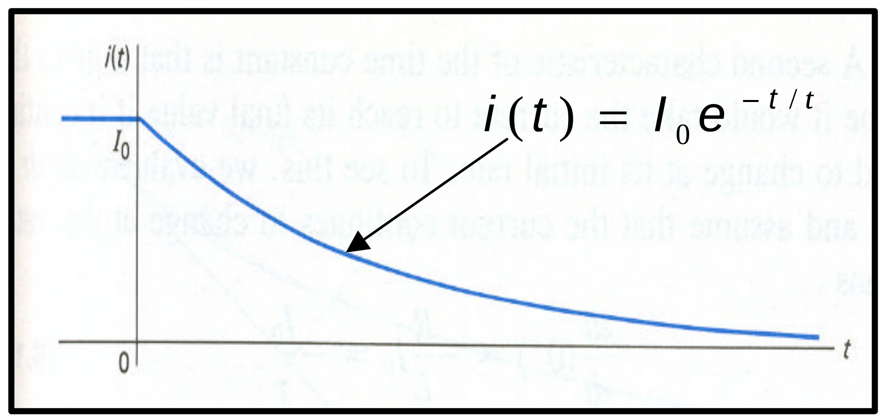




- Using KVL,

$$L\frac{di}{dt} + Ri = 0 \;\Rightarrow\; L\frac{di}{dt}dt = -\frac{R}{L}idt \;\Rightarrow\; \frac{di}{i} = -\frac{R}{L}dt$$

$$\int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dy \;\Rightarrow\; \ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$$



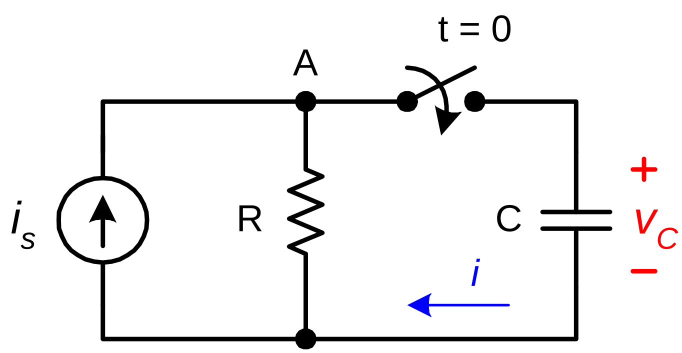
τ = time constant = L/R

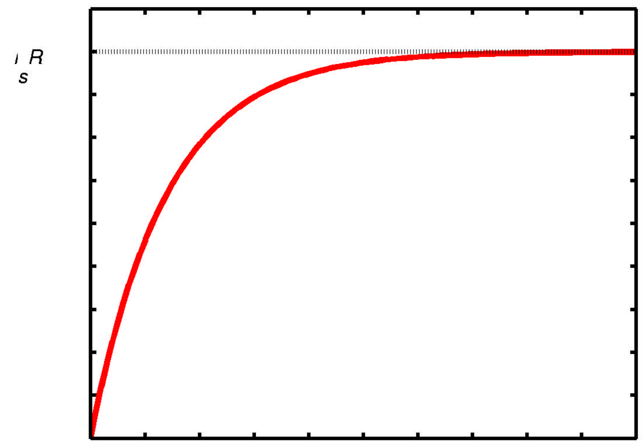
$I_0 = i(0)$

- A similar analysis can be done for an RC

Step Response of an RC Circuit

- Now, let's find the step response of an RC circuit using the following example circuit.





  – Summing the current around node A gives…

$$C\frac{dv_C}{dt}+\frac{v_C}{R}=i_s \Rightarrow \frac{dv_C}{dt}+\frac{v_C}{RC}=\frac{i_s}{C} \Rightarrow \frac{dv_C}{dt}=-\frac{1}{RC}(v_C-i_sR)$$

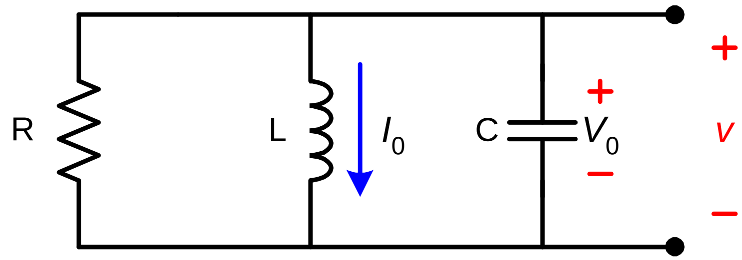
$$\Rightarrow \frac{dv_C}{v_C-i_sR}=-\frac{1}{RC}dt \Rightarrow \int_{v_C(0)}^{v_C(t)}\frac{dx}{x-i_sR}=-\frac{1}{RC}\int_0^t dt \Rightarrow \ln\frac{v_C(t)-i_sR}{v_C(0)-i_sR}=-\frac{1}{RC}t$$

$$\Rightarrow v_C(t)=i_sR+(v_C(0)-i_sR)e^{-\frac{t}{RC}} \Rightarrow v_C(t)=i_sR\left(1-e^{-\frac{t}{RC}}\right)$$

for $t > 0$ and $v_C(0) = 0$

Natural Response of Parallel RLC

- Throwing all of the components into the mix leads to a more interesting problem. Let's look at the natural response of a parallel RLC circuit.



- I_0 and V_0 are initial conditions. Solve for v (sum currents)

$$\frac{v}{R}+\frac{1}{L}\int_0^t v\,dt+I_0+C\frac{dv}{dt}=0 \Rightarrow \text{differentiate w.r.t } t \Rightarrow \frac{1}{R}\frac{dv}{dt}+\frac{v}{L}+C\frac{d^2v}{dt^2}=0$$

$$\Rightarrow \frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

General Solution for 2nd Order Differential Equation

- First, let's assume the answer is of exponential form, $v = Ae^{st}$, then we can rewrite the differential equation as

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0 \quad \Rightarrow \quad Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

- For the above equation to be a solution, *A* must equal 0 or the term in the parentheses must equal 0. *A* cannot equal zero since this means voltage is 0 for all time and that cannot be the case if there is some initial energy. Then, the characteristic equation of the differential equation is

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

because the roots of this quadratic equation determine the mathematical character of $v(t)$. The roots are…

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- So then, either of the following two are possible solutions

$$v(t) = A_1 e^{s_1 t} \qquad v(t) = A_2 e^{s_2 t}$$

and the sum is also a possible solution

$$v(t) = v_1(t) + v_2(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

then…

$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t} \qquad \frac{d^2 v}{d^2 t} = A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t}$$

and combining the equations…

$$A_1 s_1^2 e^{s_1 t}\left(s_1^2 + \frac{s_1}{RC} + \frac{1}{LC} \right) + A_2 s_2^2 e^{s_2 t}\left(s_2^2 + \frac{s_2}{RC} + \frac{1}{LC} \right) = 0$$

Each parenthetical term is 0 by definition, since s_1 and s_2 are roots of the characteristic equation. So, the natural response of the RLC circuit is…

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- The behavior of $v(t)$ depends on
  - s_1 and s_2 which depend on R, L, and C
  - Initial conditions set A_1 and A_2
- Three possible forms of behavior of the solution depending the R, L, and C
  - overdamped
  - underdamped
  - critically damped
- But first, let's define some terminology

$$s_{1,2} = -a \pm \sqrt{a^2 - w_0^2}$$ characteristic roots

$$a = \frac{1}{2RC}$$ Neper frequency

$$w_0 = \frac{1}{\sqrt{LC}}$$ Resonant radian frequency

Overdamped Response

- When the roots of the characteristic equation are real and distinct ($\alpha^2 > \omega_0^2$), the voltage response of a parallel RLC circuit is said to be overdamped. The solution is of the form

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- To solve for A_1 and A_2,

$$v(0^+) = A_1 + A_2 \quad \text{and} \quad \frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2$$

also,

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -\frac{V_0}{R} - I_0$$

- Simultaneously solve the above equations.
- Substitute values to find $v(t)$ for t>0

Underdamped Response

- When $\alpha^2 < \omega_0^2$, the roots of the characteristic equation are complex, and the response is underdamped. Rewriting the characteristic roots equation conveniently gives…

$$s_1 = -\alpha + \sqrt{-\left(\omega_0^{\ 2} - \alpha^2\right)} = -\alpha + j\sqrt{\omega_0^{\ 2} - \alpha^2} \qquad j = \sqrt{-1}$$

$$s_1 = -\alpha + j\omega_d$$
$$s_2 = -\alpha - j\omega_d \qquad \omega_d = \sqrt{\omega_0^{\ 2} - \alpha^2}$$

- To solve, we use Euler's identity (other trig identities (pdf)) → $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$
- Going back to the general form of the solution, we get

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t} \Rightarrow v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \Rightarrow v(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$\Rightarrow \quad v(t) = e^{-\alpha t}\left(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t\right)$$

$$\Rightarrow \quad v(t) = e^{-\alpha t}\left[\left(A_1 + A_2\right)\cos\omega_d t + j\left(A_1 - A_2\right)\sin\omega_d t\right]$$

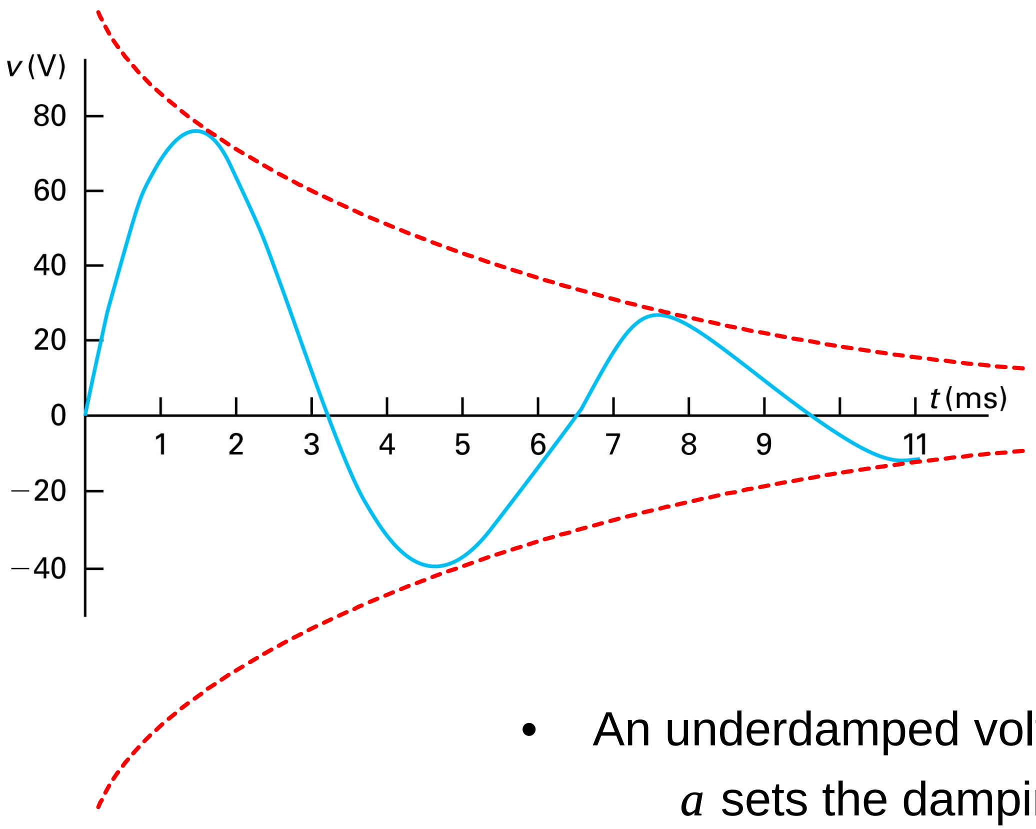
$$\Rightarrow \quad \boxed{v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t}$$

- Note that B_1 and B_2 are real for an underdamped response. This is b/c A_1 and A_2 are complex conjugates of each other. Using B_1 and B_2 just makes the calculations easier. B_1 and B_2 are a function of the initial energy stored on the capacitor and in the inductor. The two simultaneous equations to solve B_1 and B_2 are…

$$v(0^+) = V_0 = B_1 \quad \text{and} \quad \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -aB_1 + w_d B_2$$

$$\text{where} \quad i_C(0^+) + I_0 + \frac{V_0}{R} = 0 \quad \text{from KCL}$$

- Some interesting characteristics…
  - The answer is a damped sinusoid at a frequency set by w_d (where a sets how quickly the amplitude of the sinusoid diminishes)
  - As the dissipative losses decreases (i.e., R → infinity), $\alpha \to 0$ and $w_d \to w_0$
    - The response is a sinusoid at the resonant frequency



- An underdamped voltage response

  α sets the damping

  ω_d sets the frequency of oscillation

Critically Damped Response

- When $\alpha^2 = \omega_0^2$ the RLC circuit is critically damped. This is the point when the circuit is on the verge of oscillating, but w_d=0. Let's look at the solution for a critically damped response…

$$s_1 = s_2 = -a = -\frac{1}{2RC}$$

However, the solution cannot take the same form as before. The equation below *cannot* satisfy two independent initial conditions V_0 and I_0

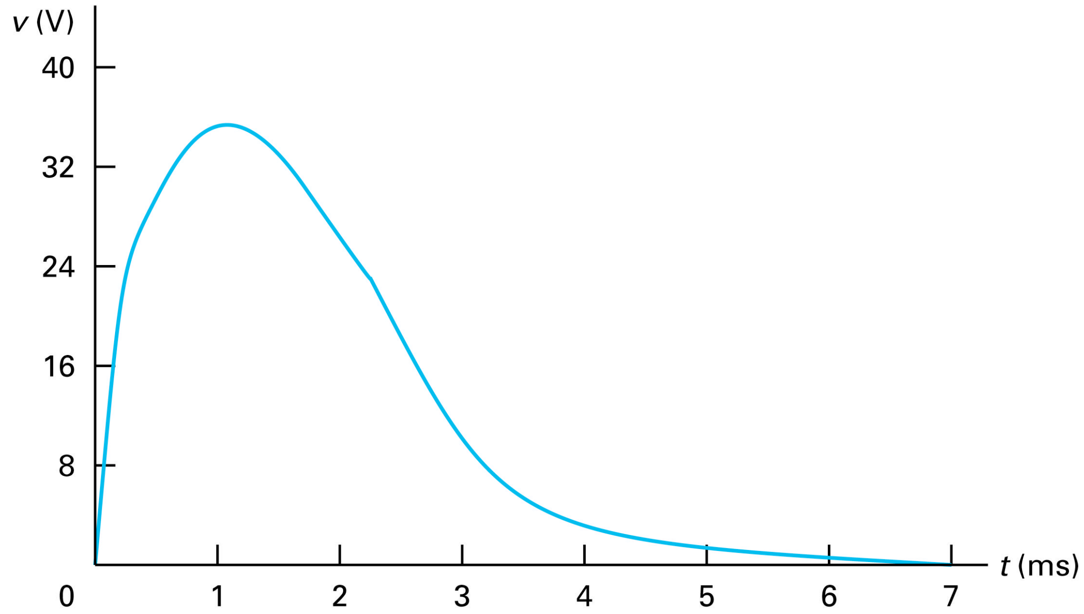
$$v = (A_1 + A_2)e^{-at} = A_0 e^{-at}$$

This is because when the roots of the characteristic equation are equal, the solution for the differential equation takes a different form…

$$v(t) = D_1 t e^{-at} + D_2 e^{-at}$$

Then, solve for D_1 and D_2

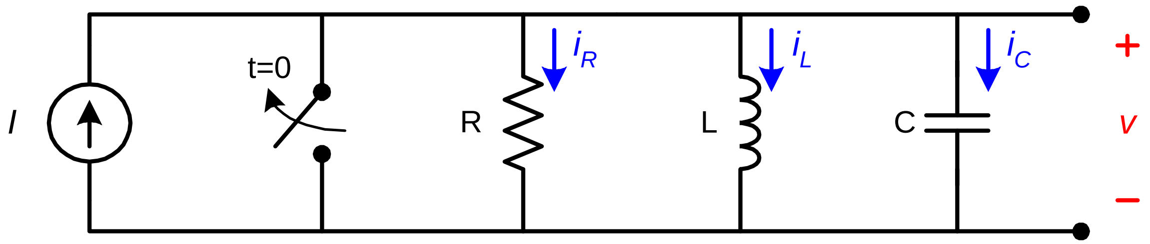
$$v(0^+) = V_0 = D_2 \qquad \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - aD_2$$



- A critically damped response
  - On verge of oscillating (but doesn't)

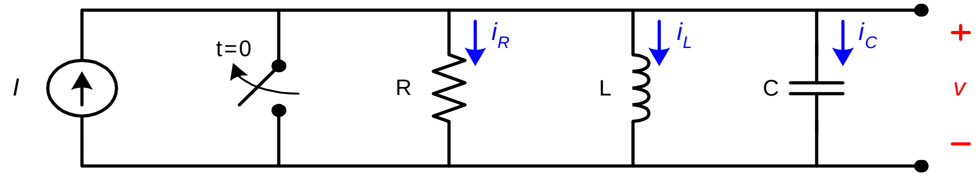
Step Response of Parallel RLC

- Now, let's see what happens when we drive the parallel RLC with a step as shown in the figure



  - KCL gives $$\frac{v}{R}+\frac{1}{L}\int_0^t v\,dt+I_0+C\frac{dv}{dt}=I$$
  - differentiate and get $$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$
  - So, the solutions for the three forms of damped responses are the same as before (in the natural response), but now you must take the *I* into account when you solve for the coefficients.

Example of Overdamped Step Response



- Assume $\alpha^2 > \omega_0^2$ and therefore the circuit is overdamped.
  So, solution is of the form (where s_1 and s_2 are real):

  $$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

  From KCL, we know that:

  $$I = i_R + i_C + i_L = \frac{v}{R} + C\frac{dv}{dt} + i_L$$

  Then, using the above solution for v, we get the following relationship:

  $$i_L = I + k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

  Let's now answer a few questions
  - What's the initial value of i_L?
    - Initially it's 0 since there is no energy stored and when the switch closes, still 0 since current through an inductor cannot change instantaneously
  - What's the initial value of v?
    - Initially 0 and when switch closes, voltage across a capacitor cannot change instantaneously

- Then, we get…

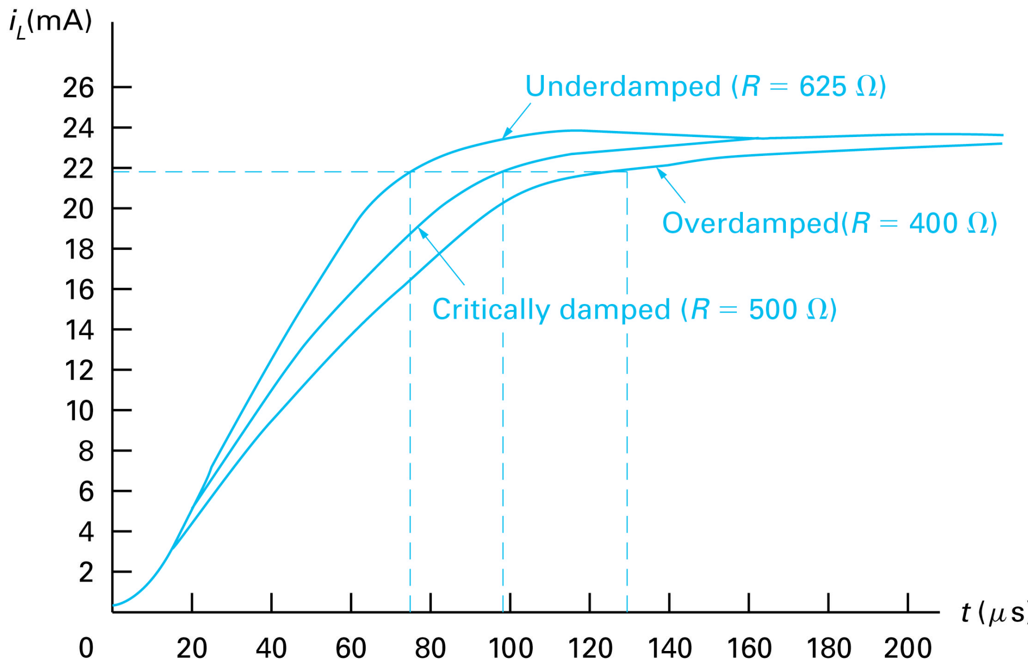
$$i_L(0) = I + k_1 + k_2 = 0 \qquad \text{and} \qquad \frac{di_L(0)}{dt} = v(0) = I + s_1 k_1 + s_2 k_2 = 0$$

  since we know s_1 and s_2, solve for k_1 and k_2.

- To solve for $v(t)$

$$v(t) = L\frac{di_L}{dt}$$

- i_L for three responses:



Solutions for RLC circuits

- One can perform a similar analysis for series RLC circuits and get a similar set of solutions.

	parallel RLC	series RLC
differential equation	$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$	$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=0$
overdamped	$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$	$i(t)=A_1e^{s_1t}+A_2e^{s_2t}$
underdamped	$v(t)=B_1e^{-at}\cos w_dt+B_2e^{-at}\sin w_dt$	$i(t)=B_1e^{-at}\cos w_dt+B_2e^{-at}\sin w_dt$
critically damped	$v(t)=D_1te^{-at}+D_2e^{-at}$	$i(t)=D_1te^{-at}+D_2e^{-at}$

Next lecture

- So far, we've developed a set of tools to analyze the response of circuits in time. Next lecture, we will pick up a set of tools that enables us to analyze circuits with respect to frequency. This requires a transformation of the equations that govern circuit operations from the time-domain to the frequency-domain. We will see how Laplace transforms enable us to do so.

- Reading:

- Supplemental Reading:
  - Nilsson Chapters 9, 12-13