# The Caldirola-Kanai Model and Its Equivalent Theories for a Damped Oscillator

Ming-Chang Huang and Ming-Chya Wu

*Department of Physics and Institute of Applied Physics, Chung Yuan Christian University, Chung Li, Taiwan 320, R.O.C.*

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Based on the Caldirola-Kanai Hamiltonian, we first construct its equivalent theories of interest. In an equivalent theory, the Hamiltonian and the relation between the two Hilbert spaces are obtained by performing a quantum canonical transformation. Then, we use the path-integral technique directly to calculate the exact propagators of the theories. The properties of the Caldirola-Kanai model, including the time evolution of the given initial wave functions and the expectation values of the physical quantities of interest, are studied by using the obtained propagators.

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## I. Introduction

The study of dissipative quantum systems, especially a damped harmonic oscillator, has a rather long history. About half a century ago, Kanai [1,2] adopted the Hamiltonian,

$$H_{CK} = e^{-\eta t}\frac{p^2}{2m_0} + e^{\eta t}\frac{1}{2}m_0\omega_0^2 x^2,\tag{1}$$

which leads exactly to the classical equation of motion of a damped harmonic oscillator,

$$\ddot{x} + \eta\dot{x} + \omega_0^2 x = 0.\tag{2}$$

This is usually referred to as the Caldirola-Kanai (CK) model. The quantum aspect of this model has been studied in a great amount of literature. In those studies some peculiarities of this model have been widely criticized and some features of it have appeared to be ambiguous [3-20]. To clarify some issues, it is important to have the exact results implied by this model. In this paper we intend to give an exact analysis of the quantum dynamics of the CK model. To achieve this goal, we construct quantum equivalent theories of the CK model and the analysis is implemented by using the CK model together with its equivalent, theories.

In classical dynamics, an equivalent theory of a model can be obtained by a canonical transformation which results in a different Hamiltonian but with the same dynamics. In this context of equivalence, one can obtain quantum equivalent theories by performing

quantum canonical transformations. Similar to the classical case, the quantum Hamiltonian of an equivalent theory may differ from that of the original model, but the quantum dynamics remains the same. Quantum canonical transformations as unitary transformations on were considered early in the development of the theory of quantum mechanics [21]. More recently, it was emphasized by Anderson [22] that the phrase "unitary equivalence" used as a synonym for "physical equivalence" might be misleading; two theories are physically equivalent if they are related by an isometric transformation, which may be either an isomorphic transformation of a Hilbert space onto itself or a linear norm-preserving isomorphic transformation from one Hilbert, space onto another. For further discussions, see Ref. [22]. We adopt this criteria for quantum equivalent theories; physical equivalence refers to the dynamical properties we study.

To obtain equivalent theories of the CK model, first we perform classical canonical transformations to obtain new Hamiltonians. Then these canonical transformations are implemented by nonunitary transformations in quantum mechanics to obtain the corresponding quantum Hamiltonians. After obtaining the quantum canonically related Hamiltonians, we use the fact that the transformed wavefunctions are square-integrable for a square-integrable wavefunction of the CK model to show that the transformations we used are indeed isometric, and hence these theories are equivalent [22]. Then we employ the path-integral technique directly to obtain the propagators. By the use of these propagators we investigate the properties of the CK model; from our results we are able to conclude that the uncertainties of position and physical momentum indeed both decay exponentially as time evolves and this decay also appears in the mean value of the mechanical energy of a damped oscillator. There are two points worthy of mention in this work: (1) From the aspect of methodology, this is the first time in the literature that the propagators of the CK Hamiltonian of Eq. (1) and its canonically related Hamiltonians were obtained by directly using the path-integral technique. (2) From the calculations performed in the CK model and its equivalent theories, we can obtain some features of the CK model in a very precise way.

This paper is organized as follows. In Section II, starting with the CK Hamiltonian, we use quantum canonical transformations to construct three other equivalent theories. These quantum canonical transformations are isometric mappings among the respective Hilbert spaces, and hence these theories are physically equivalent. In Section III, we use the path-integral method to derive the exact propagators of the CK Hamiltonian and its canonically related Hamiltonians. These propagators reduce to the propagators of a free particle, a damped free particle, and a simple harmonic oscillator, respectively, in the corresponding limiting cases. In Section IV, these propagators are used to study the time evolutions of given initial wave functions, including the Gaussian wave packets and the energy eigenstate wavefunctions of a undamped harmonic oscillator, then the uncertainties of the position and the canonical momentum, and the mean values of the Hamiltonians. Finally, we summarize our results in Section V.

## II. Equivalent theories

In this section first we use classical canonical transformations to derive three different Hamiltonians based on the CK Hamiltonian. Then these transformations are implemented

quantum mechanically to obtain equivalent theories in quantum mechanics with respect to the CK model. We find that, if we adopt the ordering that momentum operators are always put on the right hand side of position operators in the transition from a classical Hamiltonian to the corresponded quantum Hamiltonian, the Hamiltonians obtained from the quantum canonical transformations take exactly the same forms as those obtained from the classical canonical transformations.

In the CK Hamiltonian,

$$H_{CK} = e^{-\eta t}\frac{p^2}{2m_0} + e^{\eta t}\frac{1}{2}m_0\omega_0^2 x^2,\tag{3}$$

the canonical momentum $p$ is related to the physical momentum $p_k = m\dot{x}$ as

$$p = e^{\eta t}p_k.\tag{4}$$

One can perform a canonical transformation to obtain another Hamiltonian in which the canonical momentum $P_I$ equals $p_k$ and the coordinate $X_I$ exponentially expands with the evolution of time. This canonical transformation is defined by

$$P_I = e^{-\eta t}p,\quad\text{and}\quad X_I = e^{\eta t}x,\tag{5}$$

and the generating function of this transformation is given by

$$F_2^I(x, P_I, t) = e^{\eta t}xP_I.\tag{6}$$

The new Hamiltonian, which is referred to as $H_I$, reads

$$H_I = e^{\eta t}\frac{P_I^2}{2m_0} + e^{-\eta t}\frac{1}{2}m_0\omega_0^2 X_I^2 + \eta X_I P_I.\tag{7}$$

There is another interesting case which corresponds to the integral of motion of the CK Hamiltonian. This can be obtained from $H_{CK}$ by the canonical transformation defined by

$$P_{II} = e^{-\frac{1}{2}\eta t}p,\quad\text{and}\quad X_{II} = e^{\frac{1}{2}\eta t}x,\tag{8}$$

with the generating function given by

$$F_2^{II}(x, P_{II}, t) = e^{\frac{1}{2}\eta t}xP_{II}.\tag{9}$$

Then the corresponding Hamiltonian, which is referred to as $H_{II}$ is

$$H_{II} = \frac{P_{II}^2}{2m_0} + \frac{1}{2}m_0\omega_0^2 X_{II}^2 + \frac{1}{2}\eta X_{II}P_{II}.\tag{10}$$

Notice that in the context of $H_{CK}$ it is obvious from the above construction that it is impossible to make a canonical transformation so that $x$ and $p_k$ are canonically conjugate to each other. There is one more case in which the Hamiltonian takes exactly the form of a harmonic oscillator [20]. Starting with $H_{CK}$, we perform the canonical transformation defined by

$$P_{III} = e^{-\frac{1}{2}\eta t}p + e^{\frac{1}{2}\eta t}\frac{\eta}{2}m_0 x, \quad \text{and} \quad X_{III} = e^{\frac{1}{2}\eta t}x, \tag{11}$$

with the generating function given by

$$F_2^{III}(x, P_{III}, t) = e^{\frac{1}{2}\eta t}x P_{III} - e^{\eta t}\frac{\eta}{4}m_0 x^2. \tag{12}$$

Then the new Hamiltonian $H_{III}$ is

$$H_{III} = \frac{P_{III}^2}{2m_0} + \frac{1}{2}m_0\omega^2 X_{III}^2, \tag{13}$$

where $\omega^2 = \omega_0^2 - (\eta/2)^2$.

The above transformations are defined in the classical phase space which can be extended to include the time $q_0$ and its conjugate momentum $p_0$ so that the Schrödinger operator in quantum mechanics corresponds to the function

$$h = p_0 + H(q, p, q_0), \tag{14}$$

defined in this space. To construct quantum canonical transformations, we follow Anderson [22] to define the quantum phase space $\Delta$ with variables $\{q_k, p_k, k, = 0, 1\}$. The variables are members of a noncommutative algebra $\Lambda$. Then a quantum canonical transformation is a mapping $C \in \Lambda$ from $\Delta$ to $\Delta$ given by

$$C : (\{q_k, p_k; k = 0, 1\}) \longmapsto \left(\{Cq_k C^{-1}, Cp_k C^{-1}; k = 0, 1\}\right). \tag{15}$$

By the construction, the transformation $C$ preserves the canonical commutation relations and it can be either unitary or nonunitary. Note that this definition does not refer to the Hilbert space. When acting on elements of a Hilbert space, the functions $C(q, p)$ are represented by operators $\hat{C}(\hat{p}, \hat{q})$, and the transformation is from one Hilbert space to itself for the unitary case and from one Hilbert space to another for the nonunitary case. Hence, physical equivalence is proven with the existence of an isometric transformation between different Hilbert spaces [22].

To implement the classical canonical transformations quantum mechanically, we first consider the transformations generated by $F_2^I$ and $F_2^{II}$. The corresponding quantum generating function $\hat{C}(\hat{P}, \hat{X})$ is written as

$$\hat{C}(\hat{P}, \hat{X}) = \exp\left(\frac{i}{\hbar}a(t)\hat{X}\hat{P}\right). \tag{16}$$

To see that the above $\hat{C}$ is the desired function, one can use the Hausdoff-Baker formula,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{m=0}^{\infty}\frac{1}{m!}\hat{A}^m\{\hat{B}\}, \tag{17}$$

with

$$\hat{A}^m\{\hat{B}\} = [\hat{A}, \hat{A}^{m-1}\{\hat{B}\}] = [\hat{A}, [\hat{A}, \cdots [\hat{A}, [\hat{A}, \hat{B}]] \cdots]], \tag{18}$$

to show that this transformation yields

$$\hat{C}\hat{X}\hat{C}^{-1} = e^{a(t)}\hat{X}, \tag{19}$$

and

$$\hat{C}\hat{P}\hat{C}^{-1} = e^{-a(t)}\hat{P}. \tag{20}$$

Then for the classical canonical transformations defined by Eqs. (6) and (9), the corresponding quantum canonical transformations are given by Eqs. (19) and (20) with $a(t) = -\eta t$ and $-\frac{1}{2}\eta t$, respectively. With the ordering convention described before, one can see that this quantum generating function is identical to the exponential of the corresponding classical infinitesimal generator [23]. Note that this quantum canonical transformation is equivalent to a dilatation and it is one of the three generators that generate the group $SL(2, C)$ [22, 24]. This canonical transformation transforms the Schrödinger operator as

$$\hat{h}'(X, P) = \hat{C}\hat{h}\hat{C}^{-1} = \hat{p}_0 - \frac{\partial a(t)}{\partial t}\hat{C}\hat{X}\hat{P}\hat{C}^{-1} + \hat{C}\hat{H}(\hat{X}, \hat{P}, t)\hat{C}^{-1}. \tag{21}$$

This results in the new Hamiltonian

$$\hat{H}_{new}(\hat{X}, \hat{P}, t) = \hat{C}\hat{H}(\hat{X}, \hat{P}, t)\hat{C}^{-1} - \frac{\partial a(t)}{\partial t}\hat{C}\hat{X}\hat{P}\hat{C}^{-1}, \tag{22}$$

which takes exactly the same form as those given by Eqs. (7) and (10) when $a(t)$ is set to be the previous values for the transformations and the ordering convention given before is used. With the above new Hamiltonian one can see that solutions of the Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}(\hat{x}, \hat{p}, t)\Psi, \tag{23}$$

are solutions of the Schrödinger equation

$$i\hbar\frac{\partial \Psi'}{\partial t} = \hat{H}_{new}(\hat{X}, \hat{P}, t)\Psi', \tag{24}$$

with $\Psi' = \hat{C}\Psi$. With the operator $\hat{C}$ given by Eq. (16), one can show that

$$\Psi'(X) = \Psi(e^{a(t)}x). \tag{25}$$

A similar procedure can be applied to the classical canonical transformation generated by $F_2^{III}$. The corresponding quantum generating function $\hat{C}(\hat{P}, \hat{X})$ can be written as

$$\hat{C}(\hat{P}, \hat{X}) = \hat{C}_2(\hat{P}, \hat{X})\hat{C}_1(\hat{X}), \tag{26}$$

where $\hat{C}_1(\hat{X})$ and $\hat{C}_2(\hat{P}, \hat{X})$ generate their own canonical transformation. The operator $\hat{C}_1(\hat{X})$ takes the form

$$\hat{C}_1(\hat{X}) = \exp\left[ie^{\eta t}\frac{\eta}{4\hbar}m_0\hat{X}^2\right], \tag{27}$$

and the transformation it generates brings $(\hat{P}, \hat{X})$ to $(\hat{P}', \hat{X}')$ with the relation,

$$\hat{P}' = \hat{P} - e^{\eta t}\frac{\eta}{2}m_0\hat{X}, \quad \text{and} \quad \hat{X}' = \hat{X}. \tag{28}$$

This transformation brings $H_{CK}$ to the new Hamiltonian $\hat{H}'$,

$$\hat{H}' = e^{-\eta t}\frac{\hat{P}'^2}{2m_0} + e^{\eta t}\frac{1}{2}m_0\omega^2\hat{X}'^2 - \frac{1}{2}\eta\hat{X}'\hat{P}'. \tag{29}$$

The operator $\hat{C}_2(\hat{P}, \hat{X})$ takes the form

$$\hat{C}_2(\hat{P}', \hat{X}') = \exp\left[i\frac{\eta}{2\hbar}\hat{X}'\hat{P}'\right], \tag{30}$$

and it brings $(\hat{P}', \hat{X}')$ to $(\hat{P}_{III}, \hat{X}_{III})$ with the relations,

$$\hat{P}_{III} = e^{\frac{1}{2}\eta t}\hat{P}', \quad \text{and} \quad \hat{X}_{III} = e^{-\frac{1}{2}\eta t}\hat{X}'. \tag{31}$$

This transformation further brings $\hat{H}'$ to the Hamiltonian $\hat{H}_{III}$,

$$\hat{H}_{III} = \frac{\hat{P}_{III}^2}{2m_0} + \frac{1}{2}m_0\omega^2\hat{X}_{III}^2. \tag{32}$$

With the relation $\Psi' = \hat{C}\Psi$, the final transformed wave function is

$$\Psi'(\hat{X}_{III}) = \exp\left[ie^{2\eta t}\frac{\eta}{4\hbar}m_0 X_{III}^2\right]\Psi(e^{\frac{1}{2}\eta t}X_{III}). \tag{33}$$

Therefore starting with the CK model, we have constructed three theories by the use of quantum canonical transformations. To show that those theories are equivalent to the CK model, one has to show that any of the transformations we used is an isomovphism between the two Hilbert spaces. Let us consider two Hamiltonian $\hat{H}$ and $\hat{H}'$ related by a quantum canonical transformation $\hat{C}$. The inner product between two states in the Hilbert space of $\hat{H}'$ takes the form of

$$\langle\Phi|\Psi\rangle_\mu \equiv \langle\Phi|\hat{\mu}(\hat{q}, \hat{p})|\Psi\rangle_1 = \int d\Sigma\,\Phi^*(q)\hat{\mu}(\hat{q}, \hat{p})\Psi(q), \tag{34}$$

where the integration is over the spatial configuration space, and $\hat{\mu}(\hat{q}, \hat{p})$ is the measure density. The norm of states is preserved for a given canonical transformation $\hat{C}$ from $\hat{H}$ to $\hat{H}'$ if

$$\langle \Psi | \Psi \rangle_\mu = \left\langle \hat{C}^{-1} \Psi | \hat{\mu}(\hat{q}, \hat{p}) | \hat{C}^{-1} \Psi \right\rangle = \langle \Psi' | \Psi' \rangle_{\mu'}, \tag{35}$$

where the transformed measured density is

$$\hat{\mu}'(\hat{q}, \hat{p}) = \left( \hat{C}^{-1} \right)^\dagger \hat{\mu}(\hat{q}, \hat{p}) \hat{C}^{-1}. \tag{36}$$

For isomorphisms of a Hilbert space onto itself, the measure density remains unchanged, and it corresponds to a unitary transformation. But it is also possible to have isomorphisms of a Hilbert space onto another and the measure density changes with an additional factor which is not a function of momentum. For the latter, to which our cases belong, one can redefine the states by absorbing a factor from the measure density. This amounts to perform an additional canonical transformation. Therefore, to show that the canonical transformations we used are isometric is equivalent to showing that $\Psi'(X)$ is square-integrable in the new configuration space. From $\Psi'(X)$ given by Eqs. (25) and (33), by changing variable from $X's$ to $x$. in the integration $\int \Psi'^*(X) \Psi'(X) dX$ it is easy to see that $\Psi'(X)$ is square integrable for square integrable $\Psi(x)$. Hence the quantum canonical transformations given in this section are isometries between the respective Hilbert spaces and the resultant theories are equivalent.

## III. Propagators

The propagator of a system is defined as the transition amplitude from one spacetime point $(x_i, t_i)$ to another point $(x_f, t_f)$,

$$K(x_f, t_f; x_i, t_i) \equiv \langle x_f, t_f | x_i, t_i \rangle. \tag{37}$$

In this section we calculate the propagators of the system governed by the CK Hamiltonian [25] and its canonically related Hamiltonians, $H_I$ and $H_{II}$, obtained in the last section by the use of the path-integral technique with the help of the matrix method [26]. Since the Hamiltonian $H_{III}$ takes exactly the same form as a harmonic oscillator, the corresponding propagator is same as that of a harmonic oscillator.

### III-1. The Hamiltonian $H_{CK}$

Following Feynman's path-integral method, we divide the time interval into $N$ equal segments; each segment has the length $\epsilon = (t_f - t_i)/N$. After inserting the complete set of coordinate basis states at each intermediate time point,, we can write the propagator of Eq. (37) as

$$K(x_f, t_f; x_i, t_i) = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int \left[ \prod_{i=1}^{N-1} dx_i \right] \langle x_f, t_f | x_{N-1}, t_{N-1} \rangle \tag{38}$$
$$\times \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \langle x_1, t_1 | x_i, t_i \rangle.$$

It is easy to show that with the CK Hamiltonian any inner product in Eq. (38) takes the form

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \left( \frac{m_0}{2\pi i \hbar \epsilon} \right)^{1/2}$$

$$\times \exp \left\{ \frac{1}{2} \eta t_{n-1} + \frac{i\epsilon}{\eta} e^{\eta t_{n-1}} \left[ \frac{m_0}{2} \left( \frac{x_n - x_{n-1}}{\epsilon} \right)^2 - \frac{m_0}{2} \omega_0^2 \left( \frac{x_n + x_{n-1}}{2} \right) \right] \right\}. \tag{39}$$

Substituting Eq. (39) into Eq. (38), we obtain the propagator as

$$K(x_f, t_f; x_i, t_i) = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \left( \frac{m_0}{2\pi i \hbar \epsilon} \right)^{N/2} \left[ \prod_{n=1}^{N} e^{\eta t_{n-1}/2} \right] \int \left[ \prod_{i=1}^{N-1} dx_i \right]$$

$$\times \exp \left\{ \frac{i\epsilon}{\hbar} \sum_{n=1}^{N} e^{\eta t_{n-1}} \left[ \frac{m_0}{2} \left( \frac{x_n - x_{n-1}}{\epsilon} \right)^2 - \frac{m_0}{2} \omega_0^2 \left( \frac{x_n + x_{n-1}}{2} \right) \right] \right\}, \tag{40}$$

where $x_0 = x_i$ and $x_N = x_f$. From Eq. (40), we can further define the action of a damped harmonic oscillator as

$$S[x] = \int dt e^{\eta t} \left[ \frac{1}{2} m_0 \dot{x}^2 - \frac{1}{2} m_0 \omega_0^2 x^2 \right]. \tag{41}$$

The classical trajectory is the extremum of the action, and it is determined by the classical equation of motion

$$\ddot{x}_{cl}(t) + \eta \dot{x}_{cl}(t) + \omega_0^2 x_{cl}(t) = 0, \tag{42}$$

with the boundary conditions $x_{cl}(t_i) = x_i$ and $x_{cl}(t_f) = x_f$. By expanding a trajectory around the classical trajectory, $x(t) = x_{cl}(t) + \xi(t)$, we can rewrite the action as

$$S[x] = S[x_{cl}] + \int_{t_i}^{t_f} dt e^{\eta t} \frac{m_0}{2} \left[ \dot{\xi}^2(t) - \eta \xi(t) \dot{\xi}(t) - \omega_0^2 \xi^2(t) \right], \tag{43}$$

where $\xi(t)$ is the measurement of the quantum fluctuation around the classical trajectory and is subject to the boundary conditions $\xi(t_i = 0)$ and $\xi(t_f) = 0$. Consequently, the propagator of Eq. (40) can be written as

$$K(x_f, t_f; x_i, t_i) = e^{iS[x_{cl}]/\hbar} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \left( \frac{m_0}{2\pi i \hbar \epsilon} \right)^{N/2} \left[ \prod_{n=1}^{N} e^{\eta t_{n-1}/2} \right]$$

$$\times \int D\xi \exp \left[ \frac{im_0}{2\hbar} \int dt e^{\eta t} (\dot{\xi}^2 - \eta \xi \dot{\xi} - \omega_0^2 \xi^2) \right], \tag{44}$$

which is equivalent to

$$K(x_f, t_f; x_i, t_i) = e^{iS[x_{cl}]/\hbar} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \left( \frac{m_0}{2i\hbar \epsilon \pi^N} \right)^{1/2} \left[ \prod_{n=1}^{N} e^{\eta t_{n-1}/2} \right] \int \left[ \prod_{n=1}^{N-1} dy_n \right]$$

$$\times \exp \left\{ - \sum_{n=1}^{N} [a_n (y_n - y_{n-1})^2 - b_n (y_n^2 - y_{n-1}^2) - f_n (y_n + y_{n-1})^2] \right\}, \tag{45}$$

where we have set $y_n = \sqrt{m_0/(2i\hbar\epsilon)}\xi(t_n), a_n = e^{\eta t_{n-1}}, b_n = \eta\epsilon e^{\eta t_{n-1}}$, and $f_n = \frac{1}{4}(\omega_0^2\epsilon^2 e^{\eta t_{n-1}})$. Note that we also set $t_i = t_0$ and $t_f = t_N$, and the boundary conditions of $\xi$ imply that $y_0 = 0$ and $y_N = 0$. Let us define the integral $I$ by

$$I = \int \left[ \prod_{n=1}^{N-1} dy_n \right] \exp\left\{ -\sum_{n=1}^{N} \left[ a_n(y_n - y_{n-1})^2 - b_n(y_n^2 - y_{n-1}^2) - f_n(y_n + y_{n-1})^2 \right] \right\}. \quad (46)$$

To complete this integral, we employ the matrix method by identifying $y$ as a column matrix with $(N-1)$ entries,

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{pmatrix}. \quad (47)$$

Then the integral can be written as

$$I = \int dy e^{-y^T \mathbf{M} y} = \pi^{(N-1)/2}(\det \mathbf{M})^{-1/2}, \quad (48)$$

and the propagator becomes

$$K(x_f, t_f; x_i, t_i) = e^{iS[x_{cl}]/\hbar} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \left( \frac{m_0}{2\pi i\hbar\epsilon \det \mathbf{M}} \right)^{1/2} \left[ \prod_{n=1}^{N} e^{\eta t_{n-1}/2} \right]. \quad (49)$$

Here $\mathbf{M}$ is a $(N-1) \times (N-1)$ matrix and it takes the form

$$\mathbf{M} = \begin{pmatrix} c_1 & d_2 & 0 & \cdots & 0 & 0 \\ d_2 & c_2 & d_3 & \cdots & 0 & 0 \\ 0 & d_3 & c_3 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & d_{N-2} & 0 \\ 0 & 0 & \cdots & d_{N-2} & c_{N-2} & d_{N-1} \\ 0 & 0 & \cdots & 0 & d_{N-1} & c_{N-1} \end{pmatrix}, \quad (50)$$

with $c_n = a_n + a_{n+1} - b_n + b_{n+1} - f_n - f_{n+1}$ and $d_n = -(a_n + f_n)$. To make the determinant of the matrix $\mathbf{M}$ more convenient to calculate, we rescale the matrix elements as

$$c_n = c'e^{\eta t_n} \quad \text{with} \quad c' = \left( 1 - \frac{1}{4}\omega_0^2\epsilon^2 \right)(1 + e^{-\eta\epsilon}) + \eta\epsilon(1 - e^{-\eta\epsilon}),$$

$$\text{and} \quad d_n = d'e^{\eta t_n} \quad \text{with} \quad d' = -\left( 1 + \frac{1}{4}\omega_0^2\epsilon^2 \right)e^{-\eta\epsilon}, \quad (51)$$

then

$$\det \mathbf{M} = \left[ \prod_{n=2}^{N} e^{\eta t_{n-1}} \right] \det \mathbf{M}', \tag{52}$$

in which the new matrix $\mathbf{M}'$ takes the form

$$\mathbf{M}' = \begin{pmatrix} c' & d' & 0 & 0 & \cdots \\ d'e^{\eta \epsilon} & c' & d' & 0 & \cdots \\ 0 & d'e^{\eta \epsilon} & c' & d' & \cdots \\ 0 & 0 & d'e^{\eta \epsilon} & c' & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{53}$$

and the propagator is further simplified to

$$K(x_f, t_f; x_i, t_i) = e^{iS[x_{cl}]/\hbar} \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \left( \frac{m_0 e^{\eta t_i}}{2\pi i \hbar \epsilon \det \mathbf{M}'} \right)^{1/2}. \tag{54}$$

To obtain the value of $\det \mathbf{M}'$, we observe that if $I_n$ is defined as the determinant of the $n \times n$ matrix $\mathbf{M}'$, among the $I_n$'s there exists a recursion relation

$$I_{n+1} = c'I_n - d'^2 e^{\eta \epsilon} I_{n-1}, \tag{55}$$

with $I_{-1} = 0$, and $I_0 = 1$. This recursion relation can be used to formulate a differential equation. To see this, we first expand $c'$ and $d'^2 e^{\eta \epsilon}$ in terms of $\epsilon$ up to second order; the results are

$$c' = 2 - \eta \epsilon - \frac{1}{2}\epsilon^2(\omega_0^2 - \eta^2) \quad \text{and} \quad d'^2 e^{\eta \epsilon} = 1 - \eta \epsilon + \frac{1}{2}\epsilon^2(\omega_0^2 + \eta^2). \tag{56}$$

Then substituting these back into Eq. (55) yields

$$\frac{I_{n+1} - 2I_n + I_{n-1}}{\epsilon^2} = -\eta \left( \frac{I_n - I_{n-1}}{\epsilon} \right) - \omega_0^2 \left( \frac{I_n + I_{n-1}}{2} \right), \tag{57}$$

where we have neglected the term $\frac{1}{2}\eta^2(I_n - I_{n-1})$. Furthermore, if $\Phi(t_n, t_i)$ is so defined that $\lim_{\substack{\epsilon \to 0 \\ N \to \infty}} (\epsilon \det \mathbf{M}') = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \epsilon I_{N-1} = \Phi(t_f, t_i)$, then Eq. (57) can be rewritten as the differential equation

$$\frac{d^2\Phi}{dt^2} = -\eta \frac{d\Phi}{dt} - \omega_0^2 \Phi, \tag{58}$$

with the boundary conditions

$$\Phi(t, t) = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \epsilon I_0 = 0 \quad \text{and} \quad \dot{\Phi}(t, t) = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \epsilon \left( \frac{I_1 - I_0}{\epsilon} \right) = 1. \tag{59}$$

Note that $\Phi$ satisfies the same equation as the classical equation of motion of $x_{cl}$. By fitting the boundary conditions, the solution of Eq. (58) is

$$\Phi(t_f, t_i) = e^{-\eta(t_f - t_i)/2} \cdot \frac{\sin \omega(t_f - t_i)}{\omega}, \tag{60}$$

where $\omega \equiv \sqrt{\omega_0^2 - (\eta/2)^2}$. Substituting Eq. (60) for $\epsilon \det \mathbf{M'}$ in Eq. (54) yields the propagator

$$K(x_f, t_f; x_i, t_i) = \left( \frac{m_0}{2\pi i \hbar} \right)^{1/2} \left[ \frac{\omega e^{\eta(t_f + t_i)/2}}{\sin \omega(t_f - t_i)} \right]^{1/2} e^{iS[x_{cl}]/\hbar}. \tag{61}$$

To compute the classical action $S[x_{cl}]$, which can be written as

$$S[x_{cl}] = \frac{m_0}{2} \left\{ e^{\eta t_f} x_{cl}(t_f) \dot{x}_{cl}(t_f) - e^{\eta t_i} x_{cl}(t_i) \dot{x}_{cl}(t_i) \right\}, \tag{62}$$

we first solve the equation of motion, Eq. (42), to obtain the classical trajectory,

$$x_{cl}(t) = \left[ \frac{x_f e^{\eta t_f/2} \sin \omega(t - t_i) - x_i e^{\eta t_i/2} \sin \omega(t_f - t)}{\sin \omega(t_i - t_f)} \right] e^{-\eta t/2}, \tag{63}$$

Then substituting Eq. (63) into Eq. (62) yields the classical action

$$S[x_{cl}] = \frac{m_0}{2} \left( \frac{\omega e^{\eta(t_f + t_i)/2}}{\sin \omega(t_f - t_i)} \right)$$
$$\times \left\{ e^{\eta(t_f - t_i)/2} x_f^2 \left[ \cos \omega(t_f - t_i) - \frac{\eta}{2\omega} \sin \omega(t_f - t_i) \right] \right.$$
$$\left. + e^{-\eta(t_f - t_i)/2} x_i^2 \left[ \cos \omega(t_f - t_i) + \frac{\eta}{2\omega} \sin \omega(t_f - t_i) \right] - 2 x_f x_i \right\}. \tag{64}$$

By substituting Eq. (64) into Eq. (61), we obtain the exact form of the propagator for the Hamiltonian of Eq. (3).

A few remarks are made about the propagator obtained here :(1) When $\omega_0 = 0$ and $\eta \to 0$, the propagator reduces to the free particle propagator,

$$K(x_f, t_f; x_i, t_i) = \left[ \frac{m_0}{2\pi i \hbar(t_f - t_i)} \right]^{1/2} \exp \left[ \frac{im_0}{2\hbar} \frac{(x_f - x_i)^2}{t_f - t_i} \right]. \tag{65}$$

(2) When $\omega_0 = 0$, the propagator reduces to the propagator of a quasi-free particle with dissipation,

$$K(x_f, t_f; x_i, t_i) = \left( \frac{m_0}{2\pi i \hbar} \right)^{1/2} \left( \frac{\eta}{e^{-\eta t_i} - e^{-\eta t_f}} \right)^{1/2}$$
$$\times \exp \left[ \frac{im_0}{2\hbar} \left( \frac{\eta}{e^{-\eta t_i} - e^{-\eta t_f}} \right) (x_f - x_i)^2 \right]. \tag{66}$$

(3) When $\eta \to 0$, the propagator reduces to the propagator of a harmonic oscillator,

$$K(x_f, t_f; x_i, t_i) = \left(\frac{m_0}{2\pi i\hbar}\right)^{1/2} \left(\frac{\omega_0}{\sin \omega_0(t_f - t_i)}\right)^{1/2} \exp\left\{\frac{im_0}{2\hbar}\left[\frac{\omega_0}{\sin \omega_0(t_f - t_i)}\right]\right.$$
$$\left. \times \left[(x_f^2 + x_i^2)\cos \omega_0(t_f - t_i) - 2x_f x_i\right]\right\}. \tag{67}$$

### III-2. The Hamiltonian $H_I$

Next we consider the system governed by the Hamiltonian $H_I$ of Eq. (7). The corresponding action is

$$S_I[X_I] = \int dt\, e^{-\eta t}\frac{m_0}{2}[\dot{X}_I^2 - (\omega_0^2 - \eta^2)X_I^2 - 2\eta X_I\dot{X}_I]. \tag{68}$$

The classical trajectory $X_{Icl}(t)$ is determined by the equation

$$\ddot{X}_{Icl}(t) - \eta\dot{X}_{Icl}(t) + \omega_0^2 X_{Icl}(t) = 0. \tag{69}$$

Similar to the previous case, by expanding a trajectory around the classical trajectory, $X_I(t) = X_{Icl}(t) + \xi(t)$, we can rewrite the action as

$$S_I[X_I] = S_I[X_{Icl}] + \int_{t_i}^{t_f} dt\, e^{-\eta t}\frac{m_0}{2}[\dot{\xi}^2(t) + \eta\xi(t)\dot{\xi}(t) - \omega_0^2\xi^2(t)]. \tag{70}$$

Comparing Eq. (70) with Eq. (42), we can conclude that the propagator is

$$K_I(X_{If}, t_f; X_{Ii}, t_i) = \left(\frac{m_0}{2\pi i\hbar}\right)^{1/2}\left[\frac{\omega e^{-\eta(t_f+t_i)/2}}{\sin \omega(t_f - t_i)}\right]^{1/2} e^{iS_I[X_{Icl}]/\hbar}. \tag{71}$$

Here the factor before the exponential function is obtained from the factor appearing in Eq. (61) by replacing $\eta$ with $-\eta$. To compute the classical action, which can be written as

$$S_I[X_{Icl}] = \frac{m_0}{2}\left\{e^{-\eta t_f}[X_{Icl}(t_f)\dot{X}_{Icl}(t_f) - X_{Icl}^2(t_f)]\right.$$
$$\left. -e^{-\eta t_i}[X_{Icl}(t_i)\dot{X}_{Icl}(t_i) - X_{Icl}^2(t_i)]\right\}, \tag{72}$$

we see that Eq. (70) is different from Eq. (42) only by an opposite Sign in the friction coefficient, then the classical trajectories can be obtained from Eq. (63) by replacing the $x$-coordinate with the $X_I$-coordinate and $\eta$ with $-\eta$, and the result reads

$$X_{Icl}(t) = \left[\frac{X_{If}e^{-\eta t_f/2}\sin \omega(t - t_i) - X_{Ii}e^{-\eta t_i/2}\sin \omega(t_f - t)}{\sin \omega(t_i - t_f)}\right] e^{\eta t/2}. \tag{73}$$

Substituting Eq. (73) into Eq. (72) yields the classical action in the form

$$
\begin{aligned}
S_I[X_{Icl}] = \quad & \frac{m_0}{2} \left( \frac{\omega e^{-\eta(t_f+t_i)/2}}{\sin \omega(t_f - t_i)} \right) \\
& \times \left\{ e^{-\eta(t_f-t_i)/2} X_{If}^2 \left[ \cos \omega(t_f - t_i) - \frac{\eta}{2\omega} \sin \omega(t_f - t_i) \right] \right. \\
& \left. + e^{\eta(t_f-t_i)/2} X_{Ii}^2 \left[ \cos \omega(t_f - t_i) + \frac{\eta}{2\omega} \sin \omega(t_f - t_i) \right] - 2X_{If}X_{Ii} \right\}.
\end{aligned}
\tag{74}
$$

By substituting Eq. (74) into Eq. (71), we obtained the exact form of the propagator for the Hamiltonian $H_I$ of Eq. (7).

One can also use this propagator to obtain the propagators for other cases: (1) When $\omega_0 = 0$ and $\eta \to 0$, the $X_I$-coordinate is the same as the $x$-coordinate, and the propagator reduces to the same free particle propagator as the one given by Eq. (65). (2) When $\omega_0 = 0$, the propagator reduces to the propagator of a quasi-free particle with dissipation,

$$
\begin{aligned}
K_I(X_{If}, t_f; X_{Ii}, t_i) = & \left( \frac{m_0}{2\pi i \hbar} \right)^{1/2} \left( \frac{\eta}{e^{\eta t_f} - e^{\eta t_i}} \right)^{1/2} \\
& \times \exp \left[ \frac{i m_0}{2\hbar} \left( \frac{\eta}{e^{\eta t_f} - e^{\eta t_i}} \right) (X_{If} - X_{Ii})^2 \right].
\end{aligned}
\tag{75}
$$

(3) When $\eta \to 0$, the $X_i$-coordinate is the same as the $x$-coordinate, and the propagator reduces to the propagator of a harmonic oscillator given by Eq. (67).

### III-2. The Hamiltonian $H_{II}$

For the system governed by the Hamiltonian $H_{II}$ of Eq. (10), the corresponding action is

$$
S_{II}[X_{II}] = \int dt \frac{m_0}{2} [\dot{X}_{II}^2 - \omega^2 X_{II}^2 - \eta X_{II} \dot{X}_{II}].
\tag{76}
$$

The classical trajectory $X_{IIcl}(t)$ is determined by the equation

$$
\ddot{X}_{IIcl}(t) + \omega^2 X_{IIcl}(t) = 0.
\tag{77}
$$

By expanding a trajectory around the classical trajectory, $X_{II}(t) = X_{IIcl}(t) + \xi(t)$, we can rewrite the action as

$$
S_{II}[X_{II}] = S_{II}[X_{IIcl}] + \int_{t_i}^{t_f} dt \frac{m_0}{2} [\dot{\xi}^2(t) - \omega^2 \xi^2(t)].
\tag{78}
$$

The integration is the same as that for the undamped harmonic oscillator except for the replacement $\omega_0 \to \omega$, and hence the propagator is

$$
K_{II}(X_{IIf}, t_f; X_{IIi}, t_i) = \left( \frac{m_0}{2\pi i \hbar} \right)^{1/2} \left[ \frac{\omega}{\sin \omega(t_f - t_i)} \right]^{1/2} e^{iS_{II}[X_{IIcl}]/\hbar}.
\tag{79}
$$

The classical action can be rewritten as

$$S_{II}[X_{IIcl}] = \frac{m_0}{2} \int_{t_i}^{t_f} dt[\dot{X}_{IIcl}^2 - \omega^2 X_{IIcl} - \eta X_{IIcl}\dot{X}_{IIcl}]. \tag{80}$$

Substituting the classical trajectories, we obtain

$$X_{IIcl}(t) = \frac{X_{IIf}\sin\omega(t-t_i) - X_{IIi}\sin\omega(t_f-t)}{\sin\omega(t_i-t_f)}, \tag{81}$$

from Eq. (77) substrtuted into Eq. (78), we obtain the result

$$
\begin{aligned}
S_{II}[X_{IIcl}] = \frac{m_0}{2}\left(\frac{\omega}{\sin\omega(t_f-t_i)}\right)&\left\{X_{IIf}^2\left[\cos\omega(t_f-t_i) - \frac{\eta}{2\omega}\sin\omega(t_f-t_i)\right]\right.\\
&+X_{IIi}^2\left[\cos\omega(t_f-t_i) + \frac{\eta}{2\omega}\sin\omega(t_f-t_i)\right] - 2X_{IIf}X_{IIi}\left.\right\}.
\end{aligned}
\tag{82}
$$

By substituting Eq. (82) into Eq. (79), we obtain the exact form of the propagator for the Hamiltonian $H_{II}$ of Eq. (10).

One can also use this propagator to obtain the propagators for other cases: (1) When $\omega_0 = 0$ and $\eta \to 0$, the $X_{II}$-coordinate is the same as the $x$-coordinate, and the propagator reduces to the same free particle propagator as the one given by Eq. (65). (2) When $\omega_0 = 0$, the propagator reduces to the propagator of a quasi-free particle with dissipation,

$$
\begin{aligned}
K_{II}(X_{IIf}, t_f; X_{IIi}, t_i) = &\left(\frac{m_0}{2\pi i\hbar}\right)^{1/2}\left(\frac{\eta e^{\frac{\eta}{2}(t_f+t_i)}}{e^{\eta t_f} - e^{\eta t_i}}\right)^{1/2}\\
&\times \exp\left[\frac{im_0}{2\hbar}\left(\frac{\eta}{e^{\eta t_f} - e^{\eta t_i}}\right)\left(e^{\frac{\eta}{2}t_i}X_{IIf} - e^{\frac{\eta}{2}t_f}X_{IIi}\right)^2\right].
\end{aligned}
\tag{83}
$$

(3) when $\eta \to 0$, the $X_{II}$-coordinate is the same as the $x$-coordinate, and the propagator reduces to the propagator of a harmonic oscillator given by Eq. (67).

## IV. Physical properties

In this section we use the propagators obtained in the last section to study the physical properties of a damped oscillator governed by the Caldirola-Kanai Hamiltonian. We calculate the time evolution of wave functions and the mean values of the physical quantities of interst, and use the results calculated in two different Hilbert spaces to discuss the uncertainties in position and physical momentum.

### IV-1. Time evolution of wave functions

Let us first consider the motion of a wave function in the system of a damped harmonic oscillator first with the CK Hamiltonian. Then we will discuss the case when other Hamiltonians are used.

Suppose that at the initial time $t_i = 0$, we prepare a particle state described by the wave function $\Psi_0(x_i)$. Then the dissipation appears at $t = 0^+$, and it is maintained afterward. The wave function at later time $t > 0$ becomes

$$\Psi(x,t) = \int dx_i K(x,t;x_i)\Psi_0(x_i),$$ (84)

where $K(x,t;x_i)$ is the propagator of a damped harmonic oscillator given by Eq. (61). If the initial wave function is the wave packet

$$\Psi_0(x_i) = (\pi\delta^2)^{-1/4} \exp\left(-\frac{x_i^2}{2\delta^2}\right),$$ (85)

where $\delta$ is the width of the wave packet, the wave function at later time $t$ becomes

$$\Psi(x,t) = \left(\frac{\pi}{\delta^2}\right)^{1/2} \left[\frac{1}{2\delta^2} - \frac{i}{2}\frac{m_0\omega}{\hbar}\left(\frac{\cos\omega t}{\sin\omega t} + \frac{\eta}{2\omega}\right)\right]^{-1/2}$$
$$\times \left(\frac{m_0\omega e^{\eta t/2}}{2\pi i\hbar \sin\omega t}\right)^{1/2} \exp\left[-\frac{1}{2}(a-ib)x^2\right],$$ (86)

where

$$a = \frac{1}{\delta^2}e^{\eta t}\left\{1 + \left[\frac{1}{\delta^4}\left(\frac{\hbar}{m_0\omega}\right)^2 + \left(\frac{\eta}{2\omega}\right)^2 - 1\right]\sin^2\omega t + \frac{\eta}{2\omega}\sin 2\omega t\right\}^{-1},$$ (87)

and

$$b = \frac{m_0\omega}{\hbar}\frac{e^{\eta t}}{\sin\omega t}\left\{\left(\cos\omega t - \frac{\eta}{2\omega}\sin\omega t\right) - \left(\cos\omega t + \frac{\eta}{2\omega}\sin\omega t\right)\right.$$
$$\left.\times \left\{1 + \left[\frac{1}{\delta^4}\left(\frac{\hbar}{m_0\omega}\right)^2 + \left(\frac{\eta}{m_0\omega}\right)^2 - 1\right]\sin^2\omega t + \frac{\eta}{2\omega}\sin 2\omega t\right\}^{-1}\right\}.$$ (88)

Then the uncertainties of position and momentum can be computed, and the results are

$$\Delta x(t) = \frac{\delta}{\sqrt{2}}e^{-\eta t/2}\left\{1 + \left[\left(\frac{\delta_0}{\delta}\right)^4\left(\frac{\omega_0}{\omega}\right)^2 + \left(\frac{\eta}{2\omega}\right)^2 - 1\right]\sin^2\omega t + \frac{\eta}{2\omega}\sin 2\omega t\right\}^{1/2},$$ (89)

and

$$\Delta p(t) = \frac{\hbar}{\sqrt{2}\delta}e^{\eta t/2}\left\{1 + \left[\left(\frac{\delta}{\delta_0}\right)^4\left(\frac{\omega_0}{\omega}\right)^2 + \left(\frac{\eta}{2\omega}\right)^2 - 1\right]\sin^2\omega t - \frac{\eta}{2\omega}\sin 2\omega t\right\}^{1/2},$$ (90)

where $\delta_0 = \sqrt{\hbar/(m_0\omega_0)}$ is the width of the ground state wave function of an undamped oscillator. If the width of the prepared wave packet in Eq. (85) is equal to $\delta_0$, then Eqs. (89) and (90) reduce to

$$\Delta x(t) = \frac{\delta_0}{\sqrt{2}}e^{-\eta t/2}\left[1 + \frac{1}{2}\left(\frac{\eta}{\omega}\right)^2\sin^2\omega t + \frac{\eta}{2\omega}\sin 2\omega t\right]^{1/2},$$ (91)

and

$$\Delta p(t) = \frac{\hbar}{\sqrt{2}\delta_0} e^{\eta t/2} \left[ 1 + \frac{1}{2} \left( \frac{\eta}{\omega} \right)^2 \sin^2 \omega t - \frac{\eta}{2\omega} \sin 2\omega t \right]^{1/2}, \tag{92}$$

The results of Eqs. (89) and (91) are shown in Fig. 1. These results indicate that, in general, the spreading of the wave packet is suppressed by the appearance of dissipation. But for $t < 1/\eta$ and $\eta \le \omega_0$ when the initial width of a wave packet is smaller than the natural scale, $\delta_0$, the wave packet would spread. This is a relic from the case of no dissipation. For $t \gg 1/\eta$, the dissipation always dominates and the width of the wave packer decreases exponentially with small oscillations as time evolves. The product of $\Delta x(t)$ and $\Delta p(t)$ for different values of the parameters are given in Fig. 2 which shows some characteristic features: For a given $\eta$ with $\delta \ne \delta_0$, the magnitude of $\Delta x(t)\Delta p(t)$ oscillates with two different amplitudes alternatively without any damping, and the effect due to different widths is only to change the phase of the two oscillating amplitudes. For the case of $\delta = \delta_0$, the magnitude oscillates with a single amplitude which is much smaller than that of $\delta \ne \delta_0$. The effect of increasing the magnitude of $\eta$ is to increase the larger one and decrease the smaller one of the two oscillating amplitudes for $\delta \ne \delta_0$, and always to increase the amplitude for $\delta = \delta_0$. One can also see from Fig. 2 that the uncertainty principle is always satisfied during the time evolution of a wave packet.

Here we make a few remarks about the time evolution of coherent states. Coherent states of an undamped oscillator are the eigenstates of the non-Hermitian operator $a$,

$$a = \sqrt{\frac{m_0\omega_0}{2\hbar}} x + i \frac{1}{\sqrt{2m_0\hbar\omega_0}} p, \tag{93}$$

and they take the form,

$$|z\rangle = N(z) \exp(za^+)|0\rangle, \tag{94}$$

where $z$, a complex number, is the corresponding eigenvalue, $N(z) = \exp(-\frac{1}{2}|z|^2)$ is a normalization constant, $a^+$ is the adjoint of the operator $a$, and $|0\rangle$ is the ground state vector of the undamped oscillator. This set of states are "minimum uncertainty wave packets" with $\Delta x = \delta_0/\sqrt{2}$ and $\Delta p = \hbar/(\sqrt{2}\delta_0)$, and when time evolves this set of states remain the "minimum uncertainty wave packets" with the same uncertainties in position and momentum as initially. To understand the time evolution of coherent states when dissipation appears, we notice that the coherent state wave function in the coordinate representation can be written as

$$\langle x|z\rangle = \left( \frac{m_0\omega_0}{\pi\hbar} \right)^{1/4} \exp \left[ -\frac{1}{2}|z|^2 + \frac{1}{2}z^2 - \frac{m_0\omega_0}{2\hbar} \left( x - \sqrt{\frac{2\hbar}{m_0\omega_0}} z \right)^2 \right], \tag{95}$$
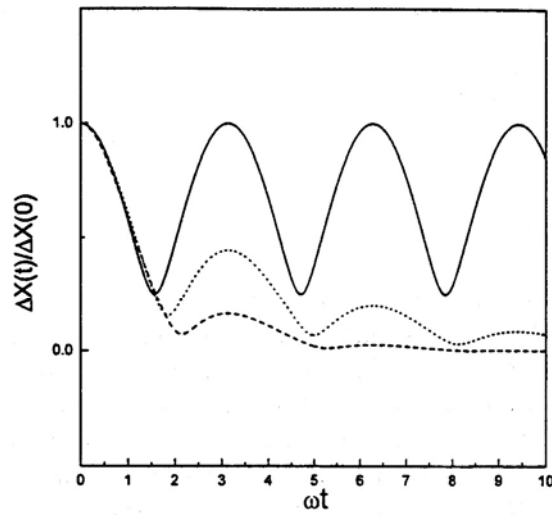
which is equivalent to a wave packet with the width $\delta_0 = \sqrt{2\hbar/(m_0\omega_0)}$ and the center moving from the origin to the complex position, $z\sqrt{2\hbar/(m_0\omega_0)}$. Because of this, we conclude that coherent states have the same time evolution features as the previously dissussed wave
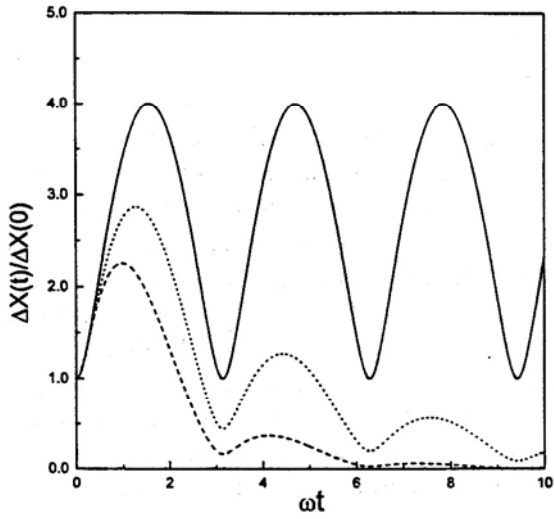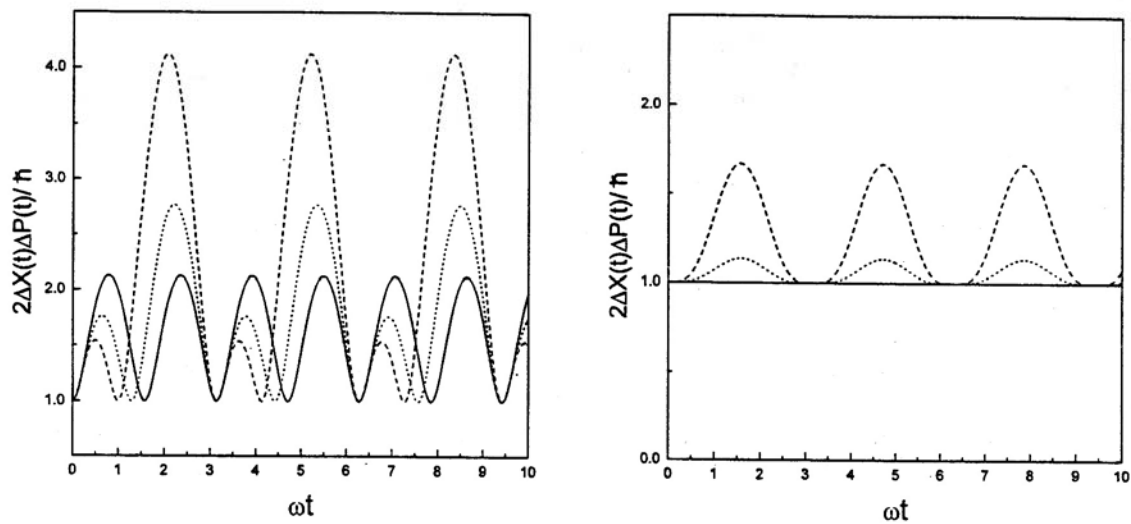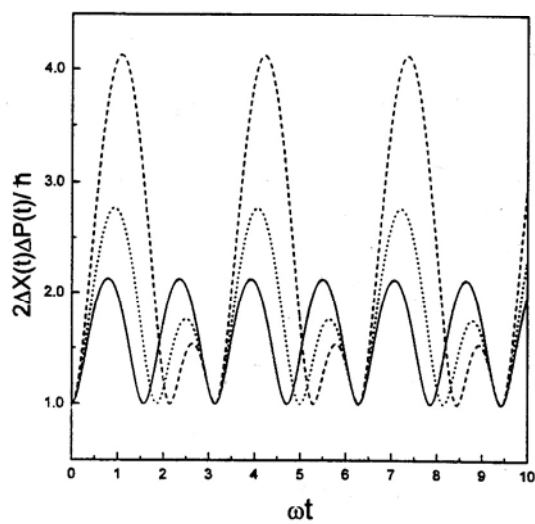
(a)
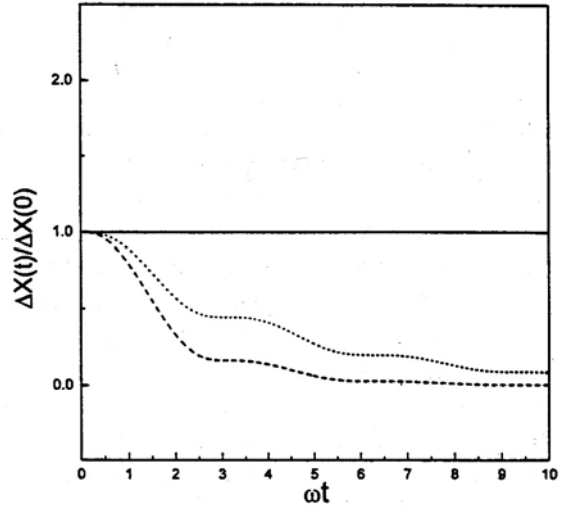
(b)

(c)

FIG. 1. The uncertainty of position as a function of time for a wave packet of initial width, (a) $\delta = 0.5\delta_0$, (b) $\delta = \delta_0$, (c) $\delta = 2\delta_0$, governed by the CK Hamiltonian with dissipation coefficients, $\eta = 0$(solid line), $0.5\omega_0$(dotted line), and $\omega_0$(broken line), respectively.

(a)



(b)



(c)

FIG. 2. The product of the uncertainties of position and momentum as a function of time for a wave packet of initial width, (a) $\delta = 0.5\delta_0$, (b) $\delta = \delta_0$, (c) $\delta = 2\delta_0$, governed by the MI Hamiltonian with dissipation coefficients, $\eta = 0$(solid line), $0.5\omega_0$(dotted line), and $\omega_0$(broken line), respectively.

packet of width $\delta_0$, and the uncertainties of position and momentum are given by Eqs. (91) and (92).

Let us consider the case that the initial state is an energy eigenstate of an undamped oscillator. The $n^{th}$ energy eigenstate of a simple harmonic oscillator is well known to be

$$\Psi_0^{(n)}(x_i) = \left(\frac{m_0\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m_0\omega_0}{\hbar}}x_i\right) \exp\left(-\frac{m_0\omega_0}{2\hbar}x_i^2\right), \qquad (96)$$

where $H_n(\zeta)$ are the Hermite polynomials of order $n$, and the corresponding eigenvalue is

$$E = \hbar\omega_0\left(n + \frac{1}{2}\right). \qquad (97)$$

If the ground state is taken as the initial state, this is the case of the wave packet defined by Eq. (85) with $\delta = \delta_0$ and the uncertainties of position and momentum are those given by Eqs. (91) and (92). In general, for the initial state being the $n^{th}$ eigenstate, the nucertainties of position and momentum in the wave function $\Psi^{(n)}(x,t)$ at the later time $t$ are

$$\Delta x^{(n)}(t) = \sqrt{n + \frac{1}{2}}\delta_0 e^{-\eta t/2}\left[1 + \frac{1}{2}\left(\frac{\eta}{\omega}\right)^2 \sin^2\omega t + \left(\frac{\eta}{2\omega}\right)\sin 2\omega t\right]^{1/2}, \qquad (98)$$

and

$$\Delta p^{(n)}(t) = \sqrt{n + \frac{1}{2}}\frac{\hbar}{\delta_0} e^{\eta t/2}\left[1 + \frac{1}{2}\left(\frac{\eta}{\omega}\right)^2 \sin^2\omega t + \left(\frac{\eta}{2\omega}\right)\sin 2\omega t\right]^{1/2}. \qquad (99)$$

All the features we obtained in the time evolution of a wave packet remain the same here.

Next we consider the case that the same initial wave functions are given but the system is governed by the Hamiltonian, $H_I$ or $H_{II}$. Similar to the previous discussions, with a given initial wave function at $t_i = 0$, one can use the propagator $K_I(X_I,t;x_i,0)$ of Eq. (71) or $K_{II}(X_{II},t;x_i,0)$ of Eq. (79) to compute the wave function $\Psi_{II}(X_I,t)$ or $\Psi_{II}(X_{II},t)$ at a later time $t > 0$. Using the wave function $\Psi_I(X_I,t)$ or $\Psi_{II}(X_{II},t)$, one can find the uncertainties of the coordinate and momentum, and the results satisfy the relations

$$\Delta X_I(t) = e^{\eta t}\Delta x(t), \quad \text{or} \quad \Delta X_{II}(t) = e^{\frac{1}{2}\eta t}\Delta x(t), \qquad (100)$$

and

$$\Delta P_{II}(t) = e^{-\eta t}\Delta p(t), \quad \text{or} \quad \Delta P_{II}(t) = e^{-\frac{1}{2}\eta t}\Delta p(t). \qquad (101)$$

These relations are the direct consequences of the quantum canonical transformations of Eqs. (19) and (20). $\Delta x(t)$ and $\Delta p(t)$ are the uncertainties of the position and the physical momentum of the damped oscillator described by the CK Hamiltonian. Here, by the calculations in two different Hilbert spaces, we show that both quantities indeed decay exponentially when time evolves as is usually expected.

### IV-2. Mean value of energy

For a given initial wave function $\Psi_0^{(n)}(x_i)$ of Eq. (96), the expectation value of the Hamiltonian $H_{CK}(t)$ with respect to the wave function $\Psi^{(n)}(x,t)$ at time $t > 0$ is given by

$$E^{(n)}(t) = \int dx \, \Psi^{(n)*}(x,t) H_{CK}(t) \Psi^{(n)}(x,t). \tag{102}$$

For the ground state wave function, $n = 0$, the above equation gives

$$E^{(0)}(t) = \frac{1}{2}\hbar\omega_0 \left(1 + \frac{\eta^2}{2\omega^2}\sin\omega t\right). \tag{103}$$

For the first excited state wave function, $n = 1$, it yields

$$E^{(1)}(t) = \frac{3}{2}\hbar\omega_0 \left(1 + \frac{\eta^2}{2\omega^2}\sin^2\omega t\right). \tag{104}$$

Continuing with the same procedure, we can conclude that the mean value of $H(t)$ at time $t > 0$ for a given initial $\Psi_0^{(n)}(x_i)$ of Eq. (96) is

$$E^{(n)}(t) = \left(n + \frac{1}{2}\right)\hbar\omega_0 \left(1 + \frac{\eta^2}{2\omega^2}\sin^2\omega t\right), \tag{105}$$

where $\omega \equiv \sqrt{\omega_0^2 - (\eta/2)^2}$. If the other two Hamiltonians, $H_I$ and $H_{II}$, are considered, the mean values of the Hamiltonians are

$$E_I^{(n)}(t) = \left(n + \frac{1}{2}\right)\hbar\omega_0 \left(1 + \frac{3\eta^2}{2\omega^2}\sin^2\omega t\right), \tag{106}$$

and

$$E_{II}^{(n)}(t) = \left(n + \frac{1}{2}\right)\hbar\omega_0 \left(1 + \frac{\eta^2}{\omega^2}\sin^2\omega t\right), \tag{107}$$

which have different spectra from Eq. (105) due to the time-dependent nonunitary transformations. These results indicate that the mean values of the Hamiltonians are the same as the eigenvalues of the Hamiltonian of an undamped oscillator with the effective frequency

$$\omega_{eff} = \omega_0 \left[1 + \left(1 - \frac{1}{\eta}\frac{\partial a}{\partial t}\right)\sin^2\omega t\right], \tag{108}$$

with $a(t)$ specified by the quantum canonical transformation, and for the classical critical damping, $\eta = 2\omega_0$, the mean values are the same as the eigenvalues of the undamped oscillator. These results also show that if $t$ is an integer multiple of the period, $T = \frac{2\pi}{\omega}$, the work done by the dissipation vanishes.

Since we have the relation

$$\langle P_I^2 \rangle = \langle p_k^2 \rangle = e^{-2\eta t}\langle p^2 \rangle, \tag{109}$$

which is a consequence of the quantum canonical transformation, from Eq. (105) we can obtain the mean value of the mechanical energy of a damped oscillator with respect to $\Psi^{(n)}(x,t)$ as

$$\left\langle \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \right\rangle_n = \left(n + \frac{1}{2}\right)\hbar\omega_0\left(1 + \frac{\eta^2}{2\omega^2}\sin\omega t\right)e^{-\eta t}. \tag{110}$$

Note that the exponential decay here is the origin of the exponential decay appearing in the uncertainties of the position and the physical momentum of a damped oscillator.

## V. Summary

We summarize our results as follows. (1) Starting with the Caldirola-Kanai Hamiltonian, we use quantum canonical transformations to obtain other three Hamiltonians. The quantum canonical transformations used in this paper are isometric mappings among different Hilbert spaces, and hence all the four theories are physically equivalent in their dynamics. (2) The exact propagators of a damped harmonic oscillator governed by these Hamiltonians are calculated explicitly. Then these propagators are used to study the time evolution of various initial state wave functions. In the CK Hamiltonian, the coordinate is the physical position of the damped oscillator, and our results show that the spreading of a wave packet in the course of time is suppressed in a way that the width of a wave packet decays exponentially with small oscillations. But for $t < 1/\eta$ and $\eta \geq \omega_0$, when the width of wave packet is smaller than $\delta_0 = \sqrt{\hbar/(m_0\omega_0)}$ the wave packet spreads, and this is a relic from the case of no dissipation. We also find a common feature in the products of the uncertainties in position and momentum, $\Delta x(t)\Delta p(t)$ and $\Delta X_i(t)\Delta P_i(t), i = I,$ or $II$. The dissipation causes these products to oscillate with two different amplitudes alternatively, except for the coherent states which are wave packets with the initial width $\delta_0$. For the coherent states, the value of the product oscillates with a single amplitude when time evolves, which is quite similar to the case of wave packets with $\delta \neq \delta_0$ in the absence of dissipation. In this respect, coherent states are quite different from an arbitrary wave packet. (3) We use another canonically related Hamiltonian of the CK Hamiltonian to study the uncertainty of the physical momentum $\Delta p_k(t)$. Our result shows that it is equal to $e^{-\eta t}\Delta p(t)$. Here $\Delta p(t)$ is the uncertainty of the canonical momentum in the Caldirola-Kanai Hamiltonian. Both the uncertainties of the position and the physical momentum of a damped oscillator decay exponentially as a function of time, and this is closely related to the exponentially decay of the mean value of the mechanical energy of a damped oscillator. (4) In the context of the CK Hamiltonian, it is impossible to make a canonical transformation so that the physical momentum is canonically conjugate to the position in the canonically transformed Hamiltonian.

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