Finite-size corrections and scaling for the triangular lattice dimer model with periodic boundary conditions

N. Sh. Izmailian*
Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan,
Yerevan Physics Institute, Alikhanian Brothers 2, 375036 Yerevan, Armenia,
and National Center of Theoretical Sciences at Taipei, Physics Division, National Taiwan University, Taipei 10617, Taiwan

K. B. Oganesyan
Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
and Yerevan Physics Institute, Alikhanian Brothers 2, 375036 Yerevan, Armenia

Ming-Chya Wu† and Chin-Kun Hu‡
Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
(Received 7 October 2005; published 24 January 2006)

We analyze the partition function of the dimer model on $\mathcal{M}\times\mathcal{N}$ triangular lattice wrapped on the torus obtained by Fendley, Moessner, and Sondhi [Phys. Rev. B **66**, 214513, (2002)]. Based on such an expression, we then extend the algorithm of Ivashkevich, Izmailian, and Hu [J. Phys. A **35**, 5543 (2002)] to derive the exact asymptotic expansion of the first and second derivatives of the logarithm of the partition function at the critical point and find that the aspect-ratio dependence of finite-size corrections and the finite-size scaling functions are sensitive to the parity of the number of lattice sites along the lattice axis.

DOI: 10.1103/PhysRevE.73.016128 PACS number(s): 05.50.+q, 75.10.−b

I. INTRODUCTION

In experiments and in numerical studies of critical systems, it is essential to take into account finite-size effects. The scaling behaviors of such corrections to the properties of infinite systems play an increasingly important role in our theoretical understanding of the critical regime of statistical systems. Therefore, in recent decades there have many investigations on finite-size scaling, finite-size corrections, and boundary effects for critical model systems [1–4].

Recently, Ivashkevich, Izmailian, and Hu (IIH) [5] proposed a systematic method to compute exact finite-size corrections to the partition functions and their derivatives of free models on torus, including the Ising model, dimer model, and Gaussian model. They found that the partition functions of all these models can be written in terms of the partition functions with twisted boundary conditions $Z_{\alpha,\beta}$ with $(\alpha,\beta)=(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$. Extending this approach, Izmailian, Oganesyan, and Hu[6] computed the finite-size corrections to the free energy for the dimer model on finite square lattices under five different boundary conditions (free, cylindrical, toroidal, Möbius strip, and the Klein bottle). They found that the aspect-ratio dependence of finite-size corrections is sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis.

In contrast to the spin models, the critical behaviors of dimer models are strongly influenced by the structure of the lattice [7]. For example, though the dimer model on the square lattice does not exhibit a phase transition [8], it is critical with algebraic decay of correlation functions [9]; the dimer model on the honeycomb lattice with anisotropic weights, which is equivalent to a five-vertex model on the square lattice, exhibits a potassium-dihydrogen-phosphate (KDP)-type singularity [10]. Moreover, the dimer model on the triangular lattice exhibits Ising-type transitions [11] precisely at the point at which it becomes the square lattice when the dimer weight in the diagonal axis becomes 0 (Fig. 1). Thus, it appears that the dimer model itself does not have a single critical behavior, but several critical behaviors associated with different classes of universality.

It has been shown explicitly [12] that the free energy per site for the dimer model on the square lattice is insensitive to the precise form of the boundary conditions in the limit of

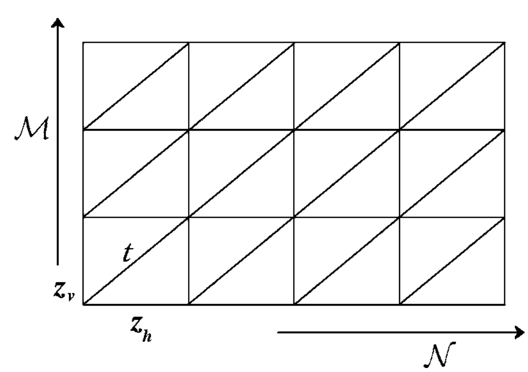

FIG. 1. Triangular lattice with dimer weights z_h in the horizontal direction, z_v in the vertical direction, and t in the diagonal direction. When $t=0$, the triangular lattice dimer model becomes the square lattice dimer model.

*Electronic address: izmailan@phys.sinica.edu.tw
†Electronic address: mcwu@phys.sinica.edu.tw
‡Electronic address: huck@phys.sinica.edu.tw

large lattices, but the finite size properties of the dimer model on the square lattice is sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis [6,13]. The above statement holds for the dimer model on the honeycomb lattice only in the case of toroidal boundary conditions. In the case of the free boundary conditions it has been proved that the free energy per site for the dimer model on the honeycomb lattice depends on the exact shape of the lattice boundary in the asymptotic limit of the large lattices [14], which actually means that in this case an infinite-size limit cannot be called a thermodynamic limit because of this lack of homogeneity. Very recently, it has been shown that the finite-size corrections of the dimer model on planar $\infty \times N$ square lattices depend crucially on the parity of N and the boundary conditions and such unusual finite-size behavior can be fully explained in the framework of the $c=-2$ logarithmic conformal field theory [15].

Our objective in this paper is to study the finite-size properties of the dimer model on the plane triangular lattice using the same techniques developed in earlier papers [5,6]. In particular, we want to know whether the behaviors of the triangular lattice dimer model near the critical point $t=0$ (Fig. 1) can be described by well defined finite-size scaling functions as in the case of the percolation [2] and Ising [3] models. As far as we know, this problem has not been studied before. Thus our study could provide a more complete picture of the critical behavior of the dimer model.

The paper is organized as follows. In Sec. II we express the exact partition function of the dimer model on the triangular lattice with periodic boundary conditions [11] as the partition functions with twisted boundary conditions $Z_{\alpha,\beta}$, where $(\alpha,\beta)=(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$. Based on such expressions and the algorithm of IIH [5], we derive the exact asymptotic expansion of the first and second derivatives of the logarithm of the partition function at the critical point ($t=t_c=0$). In Sec. III finite-size corrections are calculated. In Sec. IV we investigate the properties of the finite-size scaling functions of the lattice dimer model. We find that the triangular lattice dimer model near the critical point $t=0$ has very good finite-size scaling behavior and finite-size scaling functions are sensitive to the parity of the number of lattice sites along the lattice axis. Our results are summarized and discussed in Sec. V.

II. PARTITION FUNCTION

In the present work, we consider the dimer model on the $\mathcal{M}\times\mathcal{N}$ triangular lattice G with periodic boundary conditions. The partition function is given by

$$Z(z_h,z_v,t,\mathcal{M},\mathcal{N}) = \sum_{G'} z_h^{n_h} z_v^{n_v} t^{n_t}, \tag{1}$$

where the summation is taken over all dimer covering configurations G' on G, z_h, z_v, and t are, respectively, the dimer weight in the horizontal, vertical, and diagonal directions, and n_h, n_v, and n_t are, respectively, the number of horizontal, vertical, and diagonal dimers (Fig. 1). The dimer model on the triangular lattice undergoes a phase transition at the point $t=t_c=0$ (and likewise for z_h and z_v), where the partition function is nonanalytic as a function of t. Thus the dimer weight t plays a role similar to the reduced temperature in the Ising model. In what follows, we will set $z_h=z_v=1$.

An explicit expression for the partition function of the dimer on a $\mathcal{M}\times\mathcal{N}$ triangular lattice wrapped on torus has been obtained by Fendley, Moessner, and Sondhi [11] and can be written as

$$Z(t,\mathcal{M},\mathcal{N}) = \frac{1}{2}[G_{0,0}(t,\mathcal{M},\mathcal{N}) + G_{0,1/2}(t,\mathcal{M},\mathcal{N}) + G_{1/2,0}(t,\mathcal{M},\mathcal{N}) + G_{1/2,1/2}(t,\mathcal{M},\mathcal{N})], \tag{2}$$

where

$$G^2_{\alpha,\beta}(t,\mathcal{M},\mathcal{N}) = \prod_{m=0}^{\mathcal{M}/2-1}\prod_{n=0}^{\mathcal{N}-1} 4\left[\sin^2\frac{2\pi(n+\alpha)}{\mathcal{N}} + \sin^2\frac{2\pi(m+\beta)}{\mathcal{M}} + t^2\cos^2\left(\frac{2\pi(n+\alpha)}{\mathcal{N}} + \frac{2\pi(m+\beta)}{\mathcal{M}}\right)\right], \tag{3}$$

for even $\mathcal{M}$. Here $\alpha=0$ corresponds to the periodic boundary conditions for the underlying free fermion in the $\mathcal{N}$-direction while $\alpha=1/2$ stands for the antiperiodic boundary conditions. Similarly β controls the boundary conditions in the $\mathcal{M}$-direction.

We are interested in computing the asymptotic expansions for large $\mathcal{M}$ and $\mathcal{N}$ with a fixed aspect ratio (e.g., length to width ratio: $\xi=\mathcal{M}/\mathcal{N}$) of the free energy per site $f(t,\mathcal{M},\mathcal{N})$, internal energy $U(t,\mathcal{M},\mathcal{N})$, and specific heat $C(t,\mathcal{M},\mathcal{N})$ near the critical point $t=t_c=0$. These quantities are defined as follows:

$$f(t,\mathcal{M},\mathcal{N}) = \frac{1}{\mathcal{M}\mathcal{N}}\ln Z_{\mathcal{M},\mathcal{N}}(t), \tag{4}$$

$$U(t,\mathcal{M},\mathcal{N}) = \frac{\partial}{\partial t} f(t,\mathcal{M},\mathcal{N}), \tag{5}$$

$$C(t,\mathcal{M},\mathcal{N}) = \frac{\partial^2}{\partial t^2} f(t,\mathcal{M},\mathcal{N}). \tag{6}$$

Since the total number of sites must be even if the lattice is to be completely covered by dimers, an odd-odd case can never occur in a dimer model, and here we will consider two cases, namely an even-even (ee) case when $\mathcal{M}=2M$ and $\mathcal{N}=2N$, and an even-odd (eo) case when $\mathcal{M}=2M$ and $\mathcal{N}=2N+1$. Note, that due to the symmetry of the lattice the odd-even case (oe) ($\mathcal{M}=2M+1$, $\mathcal{N}=2N$) can be obtained from the even-odd case by a simple transformation $\xi\to 1/\xi$.

A. Dimers on $2M\times 2N$ lattices

It is easy to show that the partition function given by Eqs. (2) and (3) can be written as

$$Z(t,2M,2N) = \frac{1}{2}[Z_{0,0}^2(t,M,N) + Z_{0,1/2}^2(t,M,N) + Z_{1/2,0}^2(t,M,N) + Z_{1/2,1/2}^2(t,M,N)], \quad (7)$$

where we have introduced the partition function with twisted boundary conditions $Z_{\alpha,\beta}^2(t,M,N)$,

$$Z_{\alpha,\beta}^2(t,M,N) = \prod_{m=0}^{M-1}\prod_{n=0}^{N-1} 4\left[\sin^2\frac{\pi(n+\alpha)}{N} + \sin^2\frac{\pi(m+\beta)}{M} + t^2\cos^2\left(\frac{\pi(n+\alpha)}{N} + \frac{\pi(m+\beta)}{M}\right)\right]. \quad (8)$$

With the help of identity [16]

$$\prod_{m=0}^{M-1} 2\left[c - a\cos\left(\frac{2\pi(m+u)}{M}\right)\right] = a^M\left|2\sinh\left(\frac{M}{2}\ln[\delta_+] - i\pi u\right)\right|^2, \quad (9)$$

where c and a are real numbers such that $\delta = c/a \geqslant 1$ with $\delta_\pm = \delta \pm \sqrt{\delta^2-1}$ (so that $\delta_+\delta_- = 1$), the partition function with twisted boundary conditions $Z_{\alpha,\beta}(t,M,N)$ can be transformed into a form

$$Z_{\alpha,\beta}(t,M,N) = \prod_{n=0}^{N-1} a_{n+\alpha}(t)^{M/2}|2\sinh[M\Omega_{n+\alpha}(t) - i\pi\beta]|, \quad (10)$$

where

$$\Omega_{n+\alpha}(t) = i\phi_{n+\alpha}(t) + \operatorname{arcsinh}\sqrt{\frac{1}{2}\left(\frac{c_{n+\alpha}(N,t)}{a_{n+\alpha}(N,t)} - 1\right)}, \quad (11)$$

$$\phi_{n+\alpha}(t) = \frac{1}{2}\arctan\frac{t^2\sin\frac{2\pi(n+\alpha)}{N}}{1 - t^2\cos\frac{2\pi(n+\alpha)}{N}}, \quad (12)$$

$$a_{n+\alpha}(N,t) = \sqrt{(1-t^2)^2 + 4t^2\sin^2\frac{\pi(n+\alpha)}{N}}, \quad (13)$$

$$c_{n+\alpha}(N,t) = 1 + t^2 + 2\sin^2\frac{\pi(n+\alpha)}{N}. \quad (14)$$

At the critical point $t=0$, we have $\phi_{n+\alpha}(0)=0$, $a_{n+\alpha}(0)=1$, $c_{n+\alpha}(0)=1+2\sin^2\pi(n+\alpha)/N$ and

$$Z_{\alpha,\beta}(0,M,N) = \prod_{n=0}^{N-1}\left|2\sinh\left[M\omega\left(\frac{\pi(n+\alpha)}{N}\right) - i\pi\beta\right]\right|, \quad (15)$$

where $\omega(k) = \Omega_k(0) = \operatorname{arcsinh}(\sin k)$. Taking the derivative of Eq. (10) with respect to the variable t and then considering the limit $t \to 0$, we obtain

$$Z'_{0,0}(0,M,N) = 2M\prod_{n=1}^{N-1} 2|\sinh(M\omega)|, \quad (16)$$

and $Z'_{0,1/2}(0,M,N) = Z'_{1/2,0}(0,M,N) = Z'_{1/2,1/2}(0,M,N) = 0$.

The asymptotic expansion of $\ln Z_{\alpha,\beta}(0,M,N)$ for (α,β) equals $(0,1/2)$, $(1/2,0)$, and $(1/2,1/2)$, and $\ln Z'_{0,0}(0,M,N)$ has been given in Ref. [5] and has the forms

$$\ln Z_{\alpha,\beta}(0,M,N) = \frac{S}{\pi}\int_0^\pi \omega(x)dx + \ln\left|\frac{\theta_{\alpha,\beta}(i\xi)}{\eta(i\xi)}\right| - 2\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{\mathrm{Re}\,K_{2p+2}^{\alpha,\beta}(i\lambda\xi)}{2p+2}, \quad (17)$$

$$\ln Z'_{0,0}(0,M,N) = \frac{S}{\pi}\int_0^\pi \omega(x)dx + \frac{1}{2}\ln 4\xi S + 2\ln|\eta(i\xi)| - 2\pi\xi\sum_{p=1}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^p \frac{\Lambda_{2p}}{(2p)!}\frac{\mathrm{Re}\,K_{2p+2}^{0,0}(i\lambda\xi)}{2p+2}. \quad (18)$$

Here

$$S = MN, \quad \xi = M/N, \quad \int_0^\pi \omega(x)dx = 2\gamma, \quad \gamma = 0.915\,965\,594\ldots$$

is Catalan's constant, $\eta(\xi)$ is the Dedekind-η function, $\theta_{\alpha,\beta}(\xi)$ is the elliptic theta function, and $K_{2p+2}^{\alpha,\beta}(\tau)$ is Kronecker's double series [5]. The differential operators Λ_{2p} that have appeared in Eq. (17) can be expressed via coefficients λ_{2p} of Taylor expansion of the lattice dispersion relation $\omega(k)$,

$$\omega(k) = k\left(\lambda + \sum_{p=1}^{\infty}\frac{\lambda_{2p}}{(2p)!}k^{2p}\right) \quad (19)$$

with $\lambda=1$, $\lambda_2=-2/3$, $\lambda_4=4$, etc.,

$$\Lambda_2 = \lambda_2,$$

$$\Lambda_4 = \lambda_4 + 3\lambda_2^2\frac{\partial}{\partial\lambda},$$

$$\Lambda_6 = \lambda_6 + 15\lambda_4\lambda_2\frac{\partial}{\partial\lambda} + 15\lambda_2^3\frac{\partial^2}{\partial\lambda^2},$$

$$\vdots \quad (20)$$

Taking the second derivative of Eq. (10) with respect to variable t and then considering the limit $t \to 0$, we obtain

$$\frac{Z''_{\alpha,\beta}(0,M,N)}{Z_{\alpha,\beta}(0,M,N)} = \sum_{n=0}^{N-1} \frac{M}{2}\frac{A''(0)}{A(0)} + \mathrm{Re}\, M\sum_{n=0}^{N-1} \Omega''(0) + 2\,\mathrm{Re}\, M\sum_{n=0}^{N-1} \Omega''(0) \times\exp\{-2m[M\Omega(0)+i\pi\beta]\}, \tag{21}$$

for (α,β) equals $(0,1/2)$, $(1/2,0)$, and $(1/2,1/2)$. The expression of $\Omega''(t=0)$ has the form

$$\Omega''(0) = -2i\sin 2x + \frac{\cos^2 x + \sin^2 x\cos 2x}{\sin x\sqrt{\sin x^2 + 1}}, \tag{22}$$

with $x=\pi(n+\alpha)/N$, and its Taylor expansion is

$$\Omega''(0) = \frac{1}{x}\left[1 + \sum_{p=1}^{\infty}\frac{k_{2p}}{(2p)!}x^{2p}\right] = \frac{1}{x}\exp\left[\sum_{p=1}^{\infty}\frac{\varepsilon_{2p}}{(2p)!}x^{2p}\right], \tag{23}$$

where the coefficients ε_{2p} and k_{2p} are related to each other through the relations of moments and cumulants, $k_2=-1/3-4i$, etc. It is easy to see that the first sum in Eq. (21) turns to zero. The other terms in Eq. (21) are similar to those obtained in Ref. [5]. The only difference is that the coefficients k_{2p} in Eq. (23) are complex and have an imaginary part. Following the same procedures as in Ref. [5], we obtain

$$\frac{Z''_{\alpha,\beta}(0,M,N)}{Z_{\alpha,\beta}(0,M,N)} = \frac{2S}{\pi}\left[\frac{1}{2}\int_0^{\pi} f(x)dx + \ln\sqrt{\frac{S}{\xi}} + R_{\alpha,\beta}(\xi)\right] + \frac{1}{2}\,\mathrm{Re}\left[\left(k_2\xi\frac{\partial}{\partial(\xi)} + \lambda_2\xi^2\frac{\partial^2}{\partial(\xi)^2}\right)\ln\left|\frac{\theta_{\alpha,\beta}(i\xi)}{\eta(i\xi)}\right|\right] - \pi\xi\sum_{p=2}^{\infty}\left(\frac{\pi^2\xi}{S}\right)^{p-1}\frac{\mathrm{Re}\,\{\epsilon_{2p}K_{2p}^{\alpha,\beta}(i\lambda\xi)\}}{p(2p)!}, \tag{24}$$

where

$$f(x) = \frac{\cos^2 x + \sin^2 x\cos 2x}{\sin x\sqrt{\sin x^2 + 1}} - \frac{1}{x} - \frac{1}{\pi - x}, \tag{25}$$

$$R_{\alpha,\beta}(\xi) = -2\ln|\theta_{\alpha,\beta}(i\xi)| + C_E + 2\ln 2, \tag{26}$$

$\int_0^{\pi} f(x)dx = -2 + \ln 2 - 2\ln\pi$ and $C_E = 0.577\,215\,664\ldots$ is the Euler constant. The differential operators ϵ_{2p} that have appeared here can be expressed via the coefficients $\omega_{2p}=\varepsilon_{2p}+\lambda_{2p}(\partial)/(\partial\lambda)$ as

$$\epsilon_2 = \omega_2,$$

$$\epsilon_4 = \omega_4 + 3\omega_2^2,$$

$$\epsilon_6 = \omega_6 + 15\omega_4\omega_2 + 15\omega_2^3,$$

$$\vdots \tag{27}$$

Taking the first and the second derivative of the logarithm of the partition function given by Eq. (7) with respect to variable t and then considering the limit $t\to 0$ we obtain

$$U(0,2M,2N) = 0, \tag{28}$$

$$C(0,2M,2N) = \frac{1}{2MN}\frac{Z''_{xx}}{[Z_{0,1/2}(0,M,N)]^2 + [Z_{1/2,0}(0,M,N)]^2 + [Z_{1/2,1/2}(0,M,N)]^2}, \tag{29}$$

with

$$Z''_{xx} = (Z'_{0,0}(0,M,N))^2 + Z_{0,1/2}(0,M,N)Z''_{0,1/2}(0,M,N) + Z_{1/2,0}(0,M,N)Z''_{1/2,0}(0,M,N) + Z_{1/2,1/2}(0,M,N)Z''_{1/2,1/2}(0,M,N).$$

B. Dimers on $2M\times(2N+1)$ lattices

With the help of identities

$$\prod_{n=0}^{2N}\left[h + b\cos^2\left(f + \frac{2\pi n}{2N+1}\right) + g\sin^2\frac{2\pi n}{2N+1}\right] = \prod_{n=0}^{2N}\left[h + b\cos^2\left(f + \frac{\pi(2n+1)}{2N+1}\right) + g\sin^2\frac{\pi(2n+1)}{2N+1}\right] = \prod_{n=0}^{2N}\left[h + b\cos^2\left(f + \frac{\pi n}{2N+1}\right) + g\sin^2\frac{\pi n}{2N+1}\right], \tag{30}$$

the partition function given by Eq. (2) can be written as,

$$Z(t,\,2M,\,2N+1) = Z_{0,0}(t,\,M,\,2N+1) + Z_{0,1/2}(t,\,M,\,2N+1). \tag{31}$$

Taking the first and the second derivative of the logarithm of the partition function given by Eq. (31) with respect to variable t and then considering the limit $t \to 0$, we obtain

$$U(0,\,2M,\,2N+1) = \frac{1}{2M(2N+1)}\frac{Z'_{0,0}(0,M,2N+1)}{Z_{0,1/2}(0,M,2N+1)}, \tag{32}$$

$$C(0,\,2M,\,2N+1) = \frac{1}{2M(2N+1)}\left[\frac{Z''_{0,1/2}(0,M,2N+1)}{Z_{0,1/2}(0,M,2N+1)} - \frac{Z'_{0,0}(0,M,2N+1)^2}{Z_{0,1/2}(0,M,2N+1)^2}\right]. \tag{33}$$

Note that Eqs. (7) and (31) in the case $t=0$ coincide with the corresponding expressions of the square lattice (see Ref. [6]).

III. FINITE SIZE CORRECTIONS

After reaching this point, one can easily write down all of the terms in the exact asymptotic expansion of the free energy per site, the internal energy per site, and the specific heat per site at the critical point $t=0$. These asymptotic expansions can be written in the following form:

$$f(0,\mathcal{M},\mathcal{N}) = f_{bulk} + \sum_{p=1}^{\infty} f_p(\xi)\mathcal{S}^{-p}, \tag{34}$$

$$U(0,\mathcal{M},\mathcal{N}) = u_{bulk} + \sum_{p=1}^{\infty} u_p(\xi)\mathcal{S}^{-p+1/2}, \tag{35}$$

$$C(0,\mathcal{M},\mathcal{N}) = c_{bulk} + \sum_{p=1}^{\infty} c_p(\xi)\mathcal{S}^{-p}, \tag{36}$$

where $\mathcal{S}=\mathcal{M}\mathcal{N}$ is area of the lattice.

(a) For the dimer model on the $2M\times 2N$ lattice the expansion coefficients are:

$$f_{bulk} = \frac{\gamma}{\pi},$$

$$f_1(\xi) = \ln\frac{\theta_2^2+\theta_3^2+\theta_4^2}{2\eta^2},$$

$$f_2(\xi) = \frac{2\pi^3\xi^2}{45}\frac{\frac{7}{8}(\theta_2^{10}+\theta_3^{10}+\theta_4^{10}) + \theta_2^2\theta_3^2\theta_4^2(\theta_2^2\theta_4^2-\theta_2^2\theta_3^2-\theta_3^2\theta_4^2)}{\theta_2^2+\theta_3^2+\theta_4^2},$$

$$\vdots$$

$$u_{bulk} = 0,$$

$$u_p = 0, \quad \text{for} \quad p = 1,2,\ldots$$

$$\vdots$$

$$c_{bulk}(\xi) = \frac{1}{\pi}\ln\sqrt{\frac{\mathcal{S}}{\xi}} + \frac{1}{\pi}\left(\ln\frac{2^{3/2}}{\pi} + C_E - 1\right) + \frac{\xi}{2}R_\theta - \frac{2}{\pi}\frac{\ln\left(\theta_2^{\theta_2^2}\theta_3^{\theta_3^2}\theta_4^{\theta_4^2}\right)}{\theta_2^2+\theta_3^2+\theta_4^2}, \tag{37}$$

$$\begin{aligned} c_1(\xi) = &-\frac{\pi^2\xi^2}{18}\frac{\theta_3^4\theta_4^4(-2\theta_2^2+\theta_3^2+\theta_4^2)}{\theta_2^2+\theta_3^2+\theta_4^2} - \frac{\pi^3\xi^3}{24}\frac{\theta_2^2\theta_3^2\theta_4^2(\theta_2^{10}+\theta_3^{10}+\theta_4^{10})}{(\theta_2^2+\theta_3^2+\theta_4^2)^2} \\ &- \frac{\pi\xi}{36}\frac{(\theta_4^2-\theta_2^2)(\theta_3^4+\theta_2^2\theta_3^2+\theta_3^2\theta_4^2-\theta_2^2\theta_4^2)}{\theta_2^2+\theta_3^2+\theta_4^2} \\ &\times\left(1+8\xi\frac{\partial}{\partial\xi}\ln\theta_2\right) - \frac{\pi^2\xi^2}{6}R_\theta\ln\,\theta_3^{(\theta_2^2+\theta_4^2)(\theta_3^4+\theta_2^2\theta_4^2)} \\ &\times\theta_2^{(\theta_3^2+\theta_4^2)(\theta_3^4-3\theta_3^2\theta_4^2+\theta_4^4)}\,\theta_4^{(\theta_2^2+\theta_3^2)(\theta_2^4-3\theta_2^2\theta_3^2+\theta_3^4)}, \end{aligned} \tag{38}$$

$$\vdots$$

where $R_\theta \equiv \theta_2^2\theta_3^2\theta_4^2/(\theta_2^2+\theta_3^2+\theta_4^2)^2$,

$$\frac{\partial}{\partial\xi}\ln\,\theta_2 = -\frac{1}{2}\theta_3^2 E, \tag{39}$$

E is the elliptic integral of the second kind, $\mathcal{S}=4M\times N$, $\xi=M/N$, and $\theta_i=\theta_i(\xi)$ with $i=2,3,4$.

(b) For the dimer model on $2M\times(2N+1)$ lattice the expansion coefficients are,

$$f_{bulk} = \frac{\gamma}{\pi},$$

$$f_1(\xi) = \frac{1}{3}\ln\frac{4\theta_2\theta_3}{\theta_4^2},$$

$$f_2(\xi) = \frac{\pi^3\xi^2}{90}(14\theta_2^4\theta_3^4-\theta_4^8),$$

$$\vdots$$

$$u_{bulk} = 0,$$

$$u_1(\xi) = \frac{1}{2}\sqrt{\xi}\theta_4^2, \tag{40}$$

$$u_2(\xi) = -\frac{\pi^3\xi^{5/2}\theta_2^4\theta_3^4\theta_4^2}{24}, \tag{41}$$

$$\vdots$$

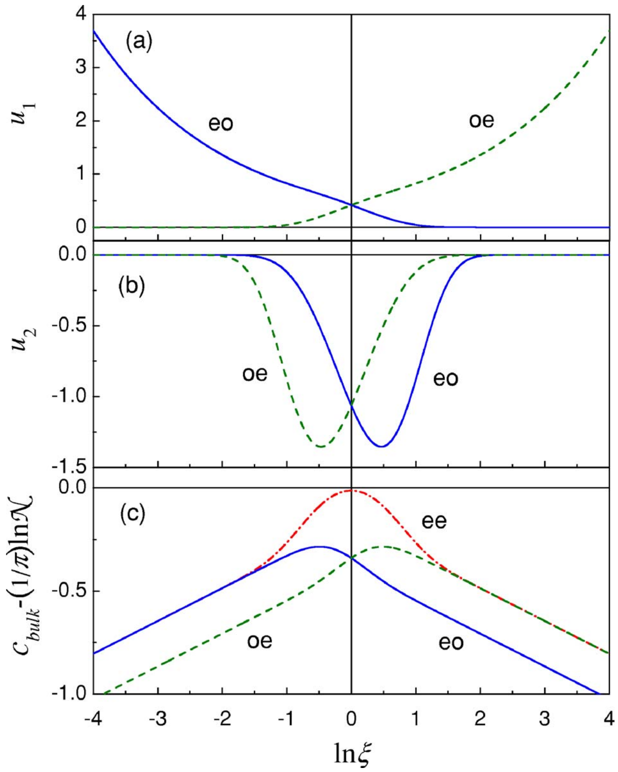

FIG. 2. (Color online) Aspect-ratio (ξ) dependence of finite-size correction terms (a) u_1, (b) u_2, and (c) $c_{bulk}-(1/\pi)\ln\mathcal{N}$, for the $2M\times 2N$ (ee), $2M\times(2N+1)$ (eo), and $(2M+1)\times 2N$ (oe) lattices. Note that eo and the oe are exchangeable by taking $\xi\to 1/\xi$, and the odd-odd case cannot occur.

$$c_{bulk}(\xi)=\frac{1}{\pi}\ln\sqrt{\frac{\mathcal{S}}{\xi}}+\frac{1}{\pi}\left(\ln\frac{2^{3/2}}{\pi}+C_E-1\right)-\frac{1}{\pi}\ln(\theta_2\theta_3)-\frac{\xi}{4}\theta_4^4, \tag{42}$$

$$c_1(\xi)=\frac{\pi^3\xi^3}{24}\theta_2^4\theta_3^4\theta_4^4+\frac{\pi^2\xi^2}{36}\theta_3^4\theta_4^4+\frac{\pi\xi(\theta_2^4+\theta_3^4)}{72}\left(1+8\xi\frac{\partial}{\partial\xi}\ln\theta_2\right). \tag{43}$$

$$\vdots$$

Here $\mathcal{S}=2M\times(2N+1)$, $\xi=2M/(2N+1)$, and $\theta_i=\theta_i(\xi)$ with $i=2,3,4$.

In Fig. 2 we plot the aspect-ratio (ξ) dependence of finite-size correction terms u_1, u_2, and $c_{bulk}-(1/\pi)\ln\mathcal{N}$, for the $2M\times 2N$ (ee), $2M\times(2N+1)$ (eo), and $(2M+1)\times 2N$ (oe) lattices. In addition, to explore different features of f and U for $\mathcal{N}$-even and -odd cases, we define

$$\delta\ln Z(t,2M,2N)=\frac{1}{2}[\ln Z(t,2M,2N-1)+\ln Z(t,2M,2N+1)]-\ln Z(t,2M,2N), \tag{44}$$

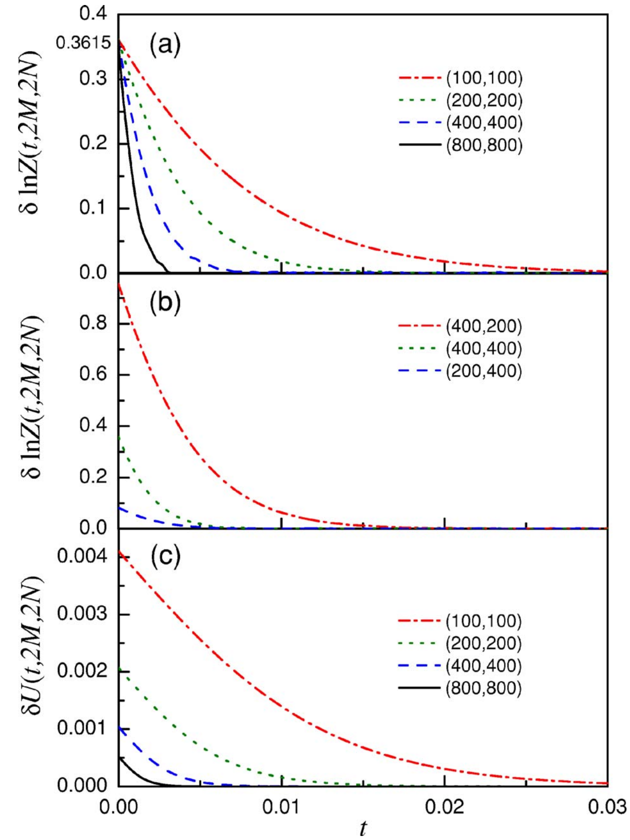

FIG. 3. (Color online) (a) $\delta\ln Z(t,\,2M,\,2N)$ as a function of t for different lattice sizes with the aspect ratio $\xi=1$. (b) $\delta\ln Z(t,\,2M,\,2N)$ as a function of t for different lattice sizes with the aspect ratio $\xi=2,1,1/2$. (c) $\delta U(t,\,2M,\,2N)$ as a function of t for different lattice sizes with the aspect ratio $\xi=1$.

$$\delta U(t,2M,2N)=U(t,2M,2N+1)-U(t,2M,2N). \tag{45}$$

The behaviors of $\delta\ln Z(t,\,2M,\,2N)$ and $\delta U(t,\,2M,\,2N)$ as functions of t for different lattice sizes are shown in Fig. 3. In Fig. 3(a), curves for different lattice sizes but with the same aspect ratio $\xi=1$ approach to a constant 0.3615 at $t=0$, which is consistent with the result of Ref. [13]. Figure 3(b) shows that $\delta\ln Z(t,\,2M,\,2N)$ is a function of ξ at the critical point $t=0$.

On the other hand, in Fig. 3(c), the amplitude of δU at $t=0$ is proportional to $1/\mathcal{N}$. Therefore, in the bulk limit, systems with even and odd $\mathcal{N}$ are not distinguishable from their internal energy U. We will further consider this situation for the specific heat C in the next section.

IV. FINITE-SIZE SCALING FUNCTIONS

The behaviors of $C(t,\mathcal{M},\mathcal{N})$ as a function of t for different lattice sizes are shown in Fig. 4. In general, $C(t,\mathcal{M},\mathcal{N})$ for $\mathcal{N}$-even and -odd cases have significant differences only near criticality, and may not be distinguishable elsewhere. Here we note that, in Fig. 4(c), the curves for lattice sizes 4000×40, 4001×40, 40×4000, and 40×4001 almost coincide into a single curve, and curves for 4000×41 and 41×4000 coincide into another curve. This implies the behav-

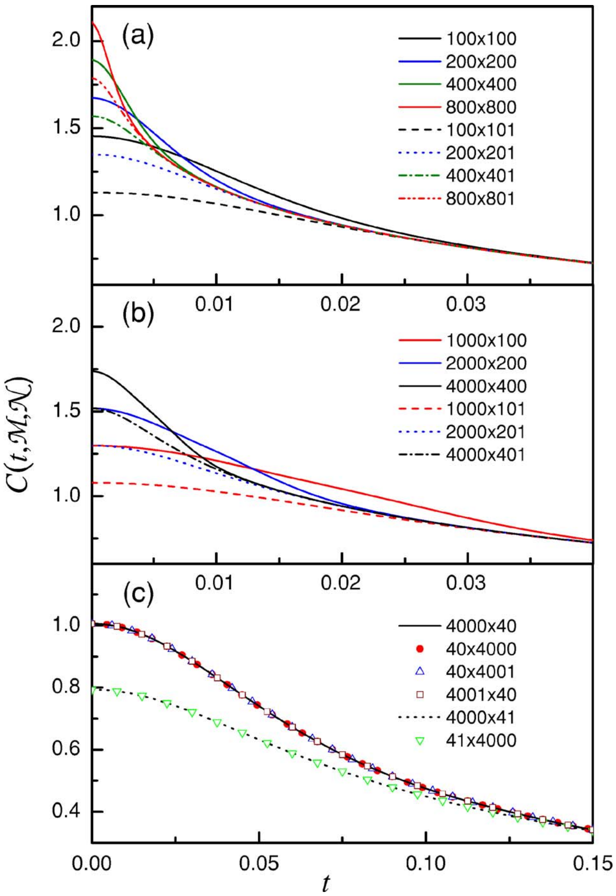

FIG. 4. (Color online) $C(t,\mathcal{M},\mathcal{N})$ as a function of t for different lattice sizes with (a) the aspect ratio $\xi=1$, (b) the aspect ratio $\xi=10$, and (c) the aspect ratio $\xi=100,0.01$.

iors of $C(t,\mathcal{M},\mathcal{N})$ near the critical point $t=0$ depend strongly on the value of $\mathcal{M},\mathcal{N}$ being even or odd for small $\mathcal{M},\mathcal{N}$. The behaviors of $C(t,2M,\mathcal{N})$ as a function of t for different lattice sizes with even and odd $\mathcal{N}$ are shown in Figs. 5(a) and 5(b), respectively.

The difference between two curves can also be calculated exactly based on the expansion of $C(t,\mathcal{M},\mathcal{N})$ at $t=0$. According to Eqs. (37) and (42), the difference between $C(0,2M,2N)$ and $C(0,2M,2N+1)$ is a finite "constant" at the bulk limit. We define this difference as

$$\begin{aligned}\delta C(\xi) &= C(0,2M\to\infty,\ 2N\to\infty)\\ &\quad - C(0,2M\to\infty,\ 2N+1\to\infty)\\ &\simeq c_{bulk}^{ee}\left(\frac{M}{N}\right) - c_{bulk}^{eo}\left(\frac{2M}{2N+1}\right),\end{aligned} \tag{46}$$

where the superscript ee denotes the ($\mathcal{M}$-even, $\mathcal{N}$-even) case and eo denotes the ($\mathcal{M}$-even, $\mathcal{N}$-odd) case. The difference δC is a function of the aspect ratio ξ, and its behavior with respect to $\ln\xi$ is shown in Fig. 6. Figure 6 provides a window for exploring the crossover behaviors of δC from $\xi \ll 1$ to $\xi \gg 1$. In particular, at $\xi=1.4444$, the difference $\delta C(\xi)$ has a maximum value, 0.3689.

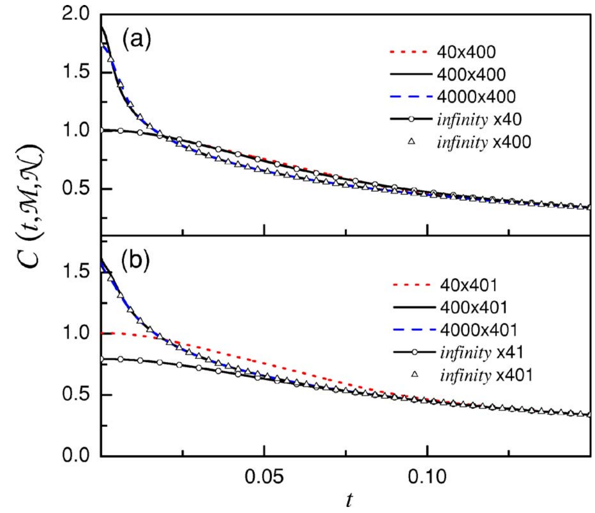

FIG. 5. (Color online) $C(t,\mathcal{M},\mathcal{N})$ as a function of t for different lattice sizes with (a) even $\mathcal{N}$, and (b) odd $\mathcal{N}$.

From the above observations, the function $C(t,\mathcal{M},\mathcal{N})$ and the dimer weight t play, respectively, roles similar to the specific heat and the reduced temperature in the Ising model. It is then interesting to investigate the properties of the finite-size scaling functions of the lattice dimer model. To do this, we propose a quantity τ, defined by $\tau=t\mathcal{S}^{1/2}$, as a metric factor for the triangular lattice dimer model. According to the scaling ansatz [17], the logarithmic divergence of $C(t,\mathcal{M},\mathcal{N})$ with respect to system size $\mathcal{S}$ at $t=0$ can be transformed to a logarithmic divergence with respect to t via the metric factor τ. Hence, in the bulk limit, the divergence of $C(t,\mathcal{M}\to\infty,\ \mathcal{N}\to\infty)$ with respect to t is logarithmic, i.e.,

$$C(t,\mathcal{M}\to\infty,\ \mathcal{N}\to\infty) \sim -\frac{1}{\pi}\ln t, \tag{47}$$

at small t (critical region). These are examined in Figs. 7(a) and 7(b), in which the curves obtained from the proposed functions $g(t)$ can fit $C(t,\mathcal{M},\mathcal{N})$ very well at the critical

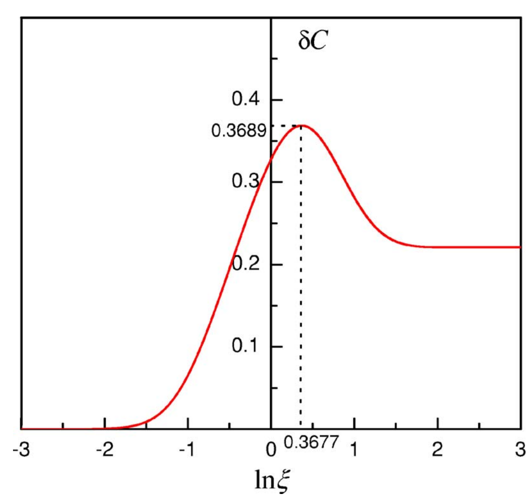

FIG. 6. (Color online) δC as a function of $\ln\xi$. The maximum of δC is 0.3689, which occurs at $\ln\xi=0.3677$ ($\xi=1.4444$).

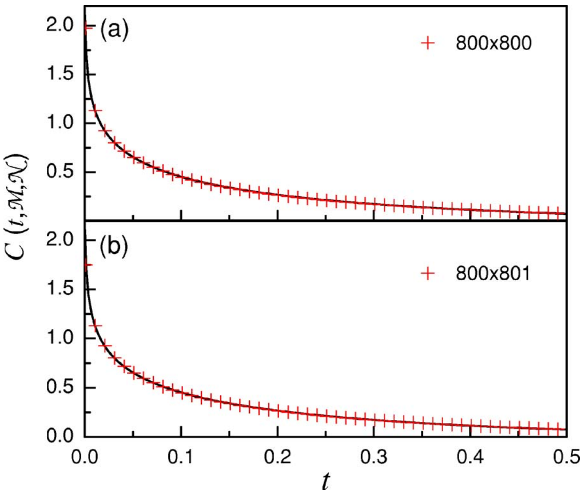

FIG. 7. (Color online) Logarithmic divergence of $C(t,\mathcal{M},\mathcal{N})$ for small t. The solid curve is obtained from (a) $g(t)=-0.3172-(1/\pi)\ln t+0.3812t-0.0830t^2-0.0112t^3+0.0006t^4$; and (b) $g(t)=-0.1592-(1/\pi)\ln t+0.3844t-0.0854t^2+0.0118t^3-0.0008t^4$.

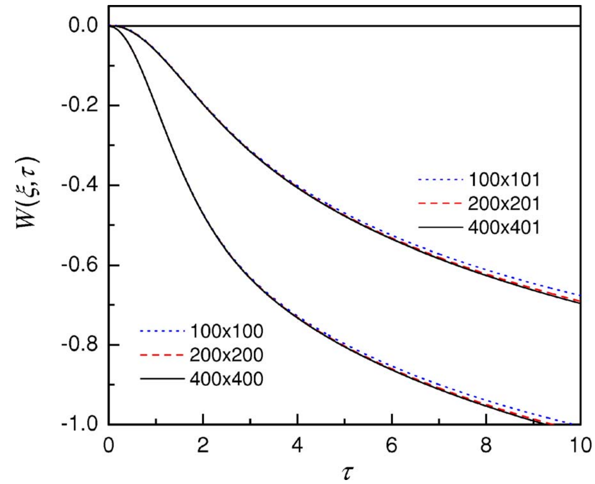

FIG. 8. (Color online) Scaling function $W(\xi,\tau)$ as a function of τ for the different lattice sizes, where $\tau=tS^{1/2}$ with S being the total number of lattice sites.

region. These confirm the fact that both of $C(t,2M,2N)$ and $C(t,2M,2N+1)$ have the same logarithmic divergence of t at the critical region.

Furthermore, according to the expansion of $C(t,\mathcal{M},\mathcal{N})$ in Eq. (36), we can define the finite-size scaling functions for $C(t,\mathcal{M},\mathcal{N})$ as in Ref. [18]. Specifically, we may define the scaling function $W(\xi,\tau)$ as

$$W(\xi,\tau) = C(t,\mathcal{M},\mathcal{N}) - c_{bulk}(\xi). \tag{48}$$

The behaviors of the scaling function $W(\xi,\tau)$ as function of τ are shown in Fig. 8. This figure shows that dimers with ($\mathcal{M}=2M$, $\mathcal{N}=2N$) and ($\mathcal{M}=2M$, $\mathcal{N}=2N+1$) have a different scaling function. In addition, the scaling function $W(\xi,\tau)$ for the two cases have very nice finite-size scaling behaviors.

V. CONCLUSIONS

In the present paper, we study the dimer model on planar $\mathcal{M}\times\mathcal{N}$ triangular lattices with periodic boundary conditions. Using the exact partition function of the dimer model on the triangular lattice with the periodic boundary condition obtained by Fendley, Moessner, and Sondhi [11] and the IIH's algorithm [5], we derive the exact asymptotic expansion of the first and second derivatives of the logarithm of the partition function in the critical point ($t=0$). We find that the aspect-ratio dependence of finite-size corrections and the finite-size scaling functions are sensitive to the parity of the number of lattice sites along the lattice axis. The free energy (specific heat) is always smaller (greater) for even-even than for even-odd lattices.

Our results inspire some interesting problems for further studies. It has been shown in Ref. [11] that the dimer model in the critical region is a lattice version of two free Majorana fermions with a mass term. Thus the partition functions and scaling behaviors of the dimer model discussed in this paper could be closely related to the behaviors of free fermions with various boundary conditions. We are working in this direction and will report the further results in a future paper.

ACKNOWLEDGMENTS

This work was supported by Academia Sinica (Taiwan) under Grant No. AS-91-TP-A02, and the National Science Council of the Republic of China (Taiwan) under Grant Nos. NSC 94-2112-M-001-014 and NSC 94-2119-M-002-001.

[1] For a review, see *Finite-size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990).

[2] C.-K. Hu, C.-Y. Lin, and J.-A. Chen, Phys. Rev. Lett. **75**, 193 (1995); **75**, 2786(E) (1995); Physica A **221**, 80 (1995); C.-K. Hu and C.-Y. Lin, Phys. Rev. Lett. **77**, 8 (1996); C.-K. Hu and F.-G. Wang, J. Korean Phys. Soc. **31**, S271 (1997); H.-P. Hsu, S. C. Lin, and C.-K. Hu, Phys. Rev. E **64**, 016127 (2001); H. Watanabe, S. Yukawa, N. Ito, and C.-K. Hu, Phys. Rev. Lett. **93**, 190601 (2004); H. Watanabe and C.-K. Hu, *ibid.* **95**, 258902 (2005).

[3] C.-K. Hu, Phys. Rev. B **29**, 5103 (1984); F. G. Wang and C.-K. Hu, Phys. Rev. E **56**, 2310 (1997); Y. Okabe, K. Kaneda, M. Kikuchi, and C.-K. Hu, *ibid.* **59**, 1585 (1999); Y. Tomita, Y. Okabe, and C.-K. Hu, *ibid.* **60**, 2716 (1999); C.-K. Hu, J.-A. Chen, and C.-Y. Lin, Physica A **266**, 27 (1999).

[4] C.-K. Hu, J. A. Chen, N. Sh. Izmailian, and P. Kleban, Phys. Rev. E **60**, 6491 (1999); N. Sh. Izmailian and C.-K. Hu, Phys.

Rev. Lett. **86**, 5160 (2001); W. T. Lu and F. Y. Wu, Phys. Rev. E **63**, 026107 (2001); K. Kaneda and Y. Okabe, Phys. Rev. Lett. **86**, 2134 (2001); N. Sh. Izmailian and C.-K. Hu, Phys. Rev. E **65**, 036103 (2002); J. Salas, J. Phys. A **34**, 1311 (2001); **35**, 1833 (2002); W. Janke and R. Kenna, Phys. Rev. B **65**, 064110 (2002); N. Sh. Izmailian, K. B. Oganesyan, and C.-K. Hu, Phys. Rev. E **65**, 056132 (2002).
[5] E. Ivashkevich, N. Sh. Izmailian, and C.-K. Hu, J. Phys. A **35**, 5543 (2002).
[6] N. Sh. Izmailian, K. B. Oganesyan, and C.-K. Hu, Phys. Rev. E **67**, 066114 (2003).
[7] For a recent review, see F. Y. Wu, e-print cond-mat/0303251.
[8] P. W. Kasteleyn, J. Math. Phys. **4**, 287 (1963).
[9] R. E. Hartwig, J. Math. Phys. **7**, 286 (1966); M. E. Fisher and J. Stephenson, Phys. Rev. **132**, 1411 (1963).
[10] F. Y. Wu, Phys. Rev. Lett. **18**, 605 (1967).
[11] P. Fendley, R. Moessner, and S. L. Sondhi, Phys. Rev. B **66**, 214513 (2002).
[12] P. W. Kasteleyn, Physica (Amsterdam) **27**, 1209 (1961).
[13] A. E. Ferdinand, J. Math. Phys. **8**, 2332 (1967).
[14] D. Grensing and G. Grensing, J. Math. Phys. **24**, 620 (1983); V. Elser, J. Phys. A **17**, 1509 (1984).
[15] N. Sh. Izmailian, V. B. Priezzhev, P. Ruelle, and C.-K. Hu, Phys. Rev. Lett. **95**, 260602 (2005).
[16] C. Nash and D. O'Connor, Ann. Phys. **273**, 72 (1999).
[17] V. Privman and J. Rudnick, J. Phys. A **19**, L1215 (1986).
[18] M.-C. Wu, C.-K. Hu, and N. Sh. Izmailian, Phys. Rev. E **67**, 065103(R) (2003).