Qualitative Analysis of Coupled Transmission Lines

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Abstract
Approach to the qualitative analysis of N-coupled transmission lines, we propose an equivalent single-line model for N coupled lines based on tri-diagonal and symmetric Toeplitz matrix characteristics of the model. Steady state error is analyzed mathematically and verified experimentally. Two cases are studied thoroughly to verify the proposition in which overall waveforms as well as individual reflections and crosstalk responses at both far and near ends are obtained from the qualitative analysis.

1. Introduction
High-speed serial links have been widely adopted in back plane interconnect. The deployment of gigabit serializer and deserializer (SerDes) poses many challenges for the design, simulation, and manufacturing of the printed circuit board (PCB). In which, transmission line analysis becomes very important for design verification and manufacture debugging. For PCB applications, crosstalk is as an important signal integrity issue as reflection.

To incorporate the crosstalk simulation into traditional circuit simulation, [1,2] use frequency domain analysis to achieve high accuracy simulation. Time domain responses can then be obtained by inverse Fourier transform. S-parameter analysis [3,4] and its model-order reduction methods [5,6] are able to achieve very accurate results. But their complexity is high. [7-9] use current and voltage dependent sources technique to produce SPICE-like model. However, they cannot separate different orders of reflections for the qualitative analysis.

In this paper, we would propose an equivalent uncoupled single-line model for the simulation of the reflection and crosstalk effects both quantitatively and qualitatively. In this mechanism, the coupled line system must be modeled as tri-diagonal and symmetric Toeplitz matrix. Hence, the crosstalk only appears between adjacent lines and all the lines are identical are assumed.

The proposed methodology is outlined as follows. First, the model for two adjacent transmission lines is derived. Then, they are decomposed into uncoupled lines mathematically. After Taylor expansion, the equivalent uncoupled model provides closed forms for the qualitative analysis of the reflection and crosstalk effects in time domain. The upper bound for the steady state error is derived. To verify the methodology, two cases are studied thoroughly. The qualitative analysis shows not only the overall waveforms but also the reflected and cross coupled noises at both ends. The negligible difference between the proposed analysis and the HSPICE simulation assure the feasibility of the methodology.

2. Transmission line model
Traditionally N-coupled quasi-TEM (transverse electromagnetic) transmission lines are modeled with distributed [R], [L], [C] and [G] elements in N by N matrices. The transmission line stretches in coordinates from 0 to l, as shown in Figure 1. The telegrapher equations are

\[
\frac{\partial v(x,t)}{\partial x} = -[L] \frac{\partial i(x,t)}{\partial t} - [R] v(x,t), \quad 0 < x < l
\]

\[
\frac{\partial i(x,t)}{\partial x} = -[C] \frac{\partial v(x,t)}{\partial t} - [G] i(x,t). \quad 0 < x < l
\]

Here \(v(x,t)\) and \(i(x,t)\) are voltage and current vectors at point \(x\) at time \(t\) as follows.

\[
v(x,t) = \begin{bmatrix} v_1(x,t) \\ v_2(x,t) \\ \vdots \\ v_N(x,t) \end{bmatrix}, \quad i(x,t) = \begin{bmatrix} i_1(x,t) \\ i_2(x,t) \\ \vdots \\ i_N(x,t) \end{bmatrix}
\]

The Laplace transforms of (1) are

\[
\frac{\partial^2 V(x,s)}{\partial x^2} = [Z][Y] V'(x,s), \quad 0 < x < l
\]

\[
\frac{\partial^2 I(x,s)}{\partial x^2} = [Y][Z] I(x,s). \quad 0 < x < l
\]
The impedance and admittance matrices in (3) are defined as
\( s = j\omega \)
\[
[Z] = s[L] + [R], \quad (4a)
\]
\[
[Y] = s[C] + [G]. \quad (4b)
\]
To uncouple the variables, a transform of basis is then introduced as follows.
\[
V(x,s) = M_v V_d(x,s) \quad 0 < x < L, \quad (5a)
\]
\[
I(x,s) = M_I I_d(x,s) \quad 0 < x < L. \quad (5b)
\]

Here, \( M_v \) and \( M_I \) are invertible matrices, \( V_d \) and \( I_d \) are the vectors of uncoupled voltages and currents respectively. Through \( V_d \) and \( I_d \), we can model a coupled system by an uncoupled one. The modified telegrapher equations are
\[
\frac{\partial^2 V_d(x,s)}{\partial x^2} = [M_v]^{-1}[Y][Z]M_v V_d(x,s), \quad (6a)
\]
\[
\frac{\partial^2 I_d(x,s)}{\partial x^2} = [M_I]^{-1}[Y][Z]M_I I_d(x,s). \quad (6b)
\]

Here, the voltage eigenvector matrix \( M_v \) and the current eigenvector \( M_I \) can be obtain from the diagonalized \([Z][Y]\) and \([Y][Z]\) matrices respectively.

From [1,2], the eigenmodes solutions for the uncoupled transmission lines can be expressed as follows
\[
V_d(x,s) = [E(x,s)]V^s_{00}(s) + [E(x,s)][V^0_{00}(s)]^{-1}V^s_{00}(s) \quad (7a)
\]
\[
I_d(x,s) = [E(x,s)]I^s_{00}(s) + [E(x,s)][I^0_{00}(s)]^{-1}I^s_{00}(s) \quad (7b)
\]

Here, \( (V^s_{00}, I^s_{00}) \) and \( (V^0_{00}, I^0_{00}) \) denote the incident and reflected traveling waves. \( (V^s_{00}, I^s_{00}) \) and \( (V^0_{00}, I^0_{00}) \) are the complex vectors of incident and reflected uncoupled voltages at \( x = 0 \). The propagation delay matrix is
\[
[E(x,s)] = diag[e^{-\gamma s},...,e^{-\gamma Ns}]. \quad (8)
\]

Each eigenvalue has a column vector in \( M_v \) and \( M_I \) as the eigenvector.

From (3) and (5), the general solutions can then be formulated as
\[
V(x,s) = [M_v][V^s_d(x,s) + V^s_I(x,s)]
\]
\[
= V^0 + V^- (x,s), \quad (9a)
\]
\[
I(x,s) = [Z_c]^{-1}[V^s(x,s) - V^-(x,s)]
\]
\[
= I^0 + I^- (x,s). \quad (9b)
\]
The characteristic impedance determines the relationship between the voltage and current on the line.
\[
[Z_c] = [M_v][Y][Z]^{-1}, \quad (10)
\]

According to the topology defined in Figure 1, the boundary conditions for forward incident and backward reflected wave at \( x = 0 \) and \( x = L \) are
\[
V^+(0,s) = [\rho_v][V^+(0,s)] + [\rho_s]V^-(0,s), \quad (11a)
\]
\[
V^-(L,s) = [\rho_L]V^+(L,s). \quad (11b)
\]
The transmission and reflection coefficients for the above equations are
\[
[\rho_v] = [Z_c][X_s + Z_c]^{-1}, \quad (12)
\]
\[
[\rho_s] = [X_s - Z_c][X_s + Z_c]^{-1}, \quad (13)
\]
\[
[\rho_L] = [X_L - Z_c][X_L + Z_c]^{-1}. \quad (14)
\]
After obtaining the transmission and reflection coefficients, let’s move back to (7). (7a) can be decomposed into two terms as follows.
\[
V^s_d(I,s) = [E(0,s)]V^s_{00}(0,s), \quad (15a)
\]
\[
V^0_d(I,s) = [E(0,s)]^{-1}V^0_{00}(0,s). \quad (15b)
\]

Form the boundary conditions (11) and (15), the forward and backward uncoupled voltages at both ends are
\[
V^s_d(0,s) = ([I_{d_{in}}] - [M_v][Y][E(0,s)][M_v]^{-1})^{-1}([M_v][Y][E(0,s)][M_v]^{-1})^{-1}[\rho_v]V^0_d(s), \quad (16a)
\]
\[
V^d(0,s) = ([I_{d_{in}}] - [M_v][Y][E(0,s)][M_v]^{-1})^{-1}([M_v][Y][E(0,s)][M_v]^{-1})^{-1}V^s_d(0,s). \quad (16b)
\]

Here, \( [I_{d_{in}}] \) is the identity matrix.

From (9a), the solution for telegrapher equations (6) with boundary conditions (15) is
\[
V(0,s) = M_v[V^s_d(0,s) + [E(I)]V^0_d(I,s)], \quad (17)
\]
\[
V(L,s) = M_v([E(I)]V^s_d(0,s) + V^0_d(l,s)). \quad (18)
\]

From here, the uncoupled voltages at both ends can be obtained from (16-18) recursively.

3. Characteristics of Toeplitz matrix

By basis transformation (5), \( M_v \) and \( M_l \) generate an uncoupled transmission line systems whose eigenvectors depend on the impedance and admittance matrices in (6). If the impedance and admittance matrices are tri-diagonal and symmetric Toeplitz matrices [7-9], the N by N capacitance and inductance matrices can be written as
\[
[C] = \begin{bmatrix}
    c_0 & -c_m & 0 & \ldots & 0 & 0 & 0 \\
    -c_m & c_0 & -c_m & \ldots & 0 & 0 & 0 \\
    0 & -c_m & c_0 & \ldots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & c_0 & -c_m & 0 \\
    0 & 0 & 0 & \ldots & -c_m & c_0 & -c_m \\
    0 & 0 & 0 & \ldots & -c_m & c_0 & -c_m \\
\end{bmatrix}, \quad (19)
\]
To decouple a coupled transmission lines, we must have a tri-diagonal and symmetric Toeplitz characteristic matrix. If all the transmission lines are identical and equally spaced, the matrix will be tri-diagonal and identical. The coupling is assumed to be significant only between immediate adjacent lines. From the above assumptions, the eigenvector matrix \([M] = [M_f] = [M_j]\) depends only on the order of the matrix. The inverse of \([M]\) is equal to its transpose. The detailed proofs are described in [7].

A Toeplitz matrix \([T]\) of order \(n\) is defined as follows.

\[
[T] = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{bmatrix}
\]

The eigenvalues of the Toeplitz matrix are

\[
\mu_i = -2\cos\left(\frac{i\pi}{N+1}\right), \quad i = 1, \ldots, N.
\]

From (19,20), \((c_s, l_s)\) represent self capacitance and inductance, and \((c_{ms}, l_{ms})\) the mutual capacitance and inductance. The eigenvalues of \([C]\) and \([L]\) are

\[
\lambda_{Cl} = c_s - c_{ms}\mu_i, \quad i = 1, \ldots, N.
\]

\[
\lambda_{Li} = l_s + l_{ms}\mu_i, \quad i = 1, \ldots, N.
\]

For the tri-diagonal and symmetric Toeplitz matrices \([Z]\) and \([Y]\) and under the lossless assumption, the eigenvalues of \([Z\:Y\:Y\:Z]\) can be obtained as follows.

\[
\gamma_i = s \cdot \sqrt{\lambda_{Cl} \cdot \lambda_{Li}}, \quad i = 1, \ldots, N.
\]

The elements of matrix \([M]\) can be computed recursively by [7-9]

\[
m_{ij} = \frac{\phi_{i+1}(\mu_j)}{\delta_j},
\]

\[
\phi_i(\mu) = \phi_{i+1}(\mu) - \phi_{i-1}(\mu), \quad i \geq 2,
\]

\[
\phi_0(\mu) = 1, \quad \phi_1(\mu) = \mu,
\]

\[
\delta_j^2 = \sum_{i=1}^{N} (\phi_{i+1}(\mu_j))^2.
\]

For example, for a coupled line pair, the eigenvectors of its tri-diagonal and symmetric Toeplitz matrix is

\[
[M] = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}.
\]

4. Equivalent single transmission line model

Equations (17,18) are the general solutions for a set of coupled lines. Now, we will study a special case with only one transmission line as the references. Here, \([L]\) and \([C]\) are \(1\times1\) matrices. The single line model can be derived as follows.

\[
V(0,s) = \frac{1 + \rho_1e^{-\gamma L}}{1 - \rho_1^2e^{-2\gamma L}} \tau_s V(s),
\]

\[
V(l,s) = \frac{(1 + \rho_1)e^{-\gamma L}}{1 - \rho_1^2e^{-2\gamma L}} \tau_s V(s).
\]

The Taylor series expansions of (31) and (32) in \(e^{-2\gamma L}\) become

\[
V(0,s) = \tau_s V(s) + \rho_1(1 + \rho_1)e^{-\gamma L},
\]

\[
V(l,s) = \tau_s V(s)(1 + \rho_1)e^{-\gamma L} + \rho_2\rho_1(1 + \rho_1)e^{-\gamma L} + \rho_3\rho_2\rho_1(1 + \rho_1)e^{-\gamma L} + 
\]

Here, \(e^{-2\gamma L}\) represents both propagation delay and reflection coefficient. For N-coupled lines analysis, (12-14) represent transmission and reflection coefficients matrices. If \([X_1, X_2]\) are identity matrices and \([Z, Y]\) are tri-diagonal symmetric Toeplitz matrices, the transmission and reflection coefficients have the same eigenvector \([M]\).

Proof: Form above conditions, the characteristic impedance matrix can be written as

\[
[Z_c] = [M][\gamma]^{-1}[M]^t[Z].
\]

We define \([Z_{cd}]\) matrix as

\[
\]

Here, \([\gamma]^{-1}\) is a diagonal matrix and \([M]\) is the eigenvectors of impedance matrix \([Z]\).

Similarly, the new coefficient matrices can be defined as

\[
[r_{cd}] = [M]^{-1} r_c[M],
\]

\[
[r_{cd}] = [Z_c][M][Z_c]^{-1}.
\]

Since all the terms are diagonal matrices, we have completed the proof.
the lines at each ends must be identical as explained earlier. Therefore,

\[
[X_S] = \begin{bmatrix} x_S & 0 \\ 0 & x_S \end{bmatrix}, \quad [X_L] = \begin{bmatrix} x_L & 0 \\ 0 & x_L \end{bmatrix}.
\] (40)

In a two-line system, those coefficients for diagonal matrices (36-39) can be defined as

\[
[Z_{ad}] = \begin{bmatrix} z_{ad1} & 0 \\ 0 & z_{ad2} \end{bmatrix}, \quad [\tau_{ad}] = \begin{bmatrix} \tau_{ad1} & 0 \\ 0 & \tau_{ad2} \end{bmatrix},
\]

\[
[\rho_{Sdl}] = \begin{bmatrix} \rho_{Sdl1} & 0 \\ 0 & \rho_{Sdl2} \end{bmatrix}, \quad [\rho_{Ld}] = \begin{bmatrix} \rho_{Ld1} & 0 \\ 0 & \rho_{Ld2} \end{bmatrix}.
\] (41) (42) (43) (44)

With (41-44), (17) and (18) are rewritten as

\[
V(0, s) = [M]V_{ad}(0, s) + [E(1)]p_{ad}(l, s)[M]^{-1}V_s(s),
\]

\[
V(l, s) = [M][E(1)]v_{ad}(0, s) + v_{ad}(l, s)[M]^{-1}V_s(s).
\] (45) (46)

\[
u_{ad}(s) = [I_f - [Sd][\rho_{Sdl}][E(2)])^{-1}[\tau_{ad}]
\]

\[
= \begin{bmatrix} \tau_{ad1} & 0 \\ 0 & \tau_{ad2} \end{bmatrix}
\]

\[
u_{ad}(l, s) = [\rho_{Ld}][E(1)]v_{ad}(0, s)
\]

\[
= \begin{bmatrix} 1 - \rho_{Sdl1}\rho_{Ld1}e^{-2\gamma_1l} & 0 \\ 0 & 1 - \rho_{Sdl2}\rho_{Ld2}e^{-2\gamma_2l} \end{bmatrix}
\] (47) (48)

Here \((v_{ad}(0, s), v_{ad}(l, s))\) are the equivalent coefficients for the single line model. (45,46) transforms the basis \([M]\) into a coupled system. Note that, \([M]\) is a matrix of constant eigenvector (26-29).

If only one input source voltage \(V_{s1}(s)\) is considered, the input source voltage matrix is

\[
[V_s] = \begin{bmatrix} V_{s1}(s) \\ 0 \end{bmatrix}.
\] (49)

After applying (49) to (45-48), the closed forms for the near end are:

\[
V_1(0, s) = m_{11} \cdot m_{11} V_{ad1}(0, s) + m_{12} \cdot m_{12} V_{ad2}(0, s)
\]

\[
= \frac{1}{2} \left( \frac{1 + \rho_{Ld1}e^{-2\gamma_1l}}{1 - \rho_{Sdl1}\rho_{Ld1}e^{-2\gamma_1l}} \right) \tau_{ad1} + \frac{1 + \rho_{Ld2}e^{-2\gamma_2l}}{1 - \rho_{Sdl2}\rho_{Ld2}e^{-2\gamma_2l}} \tau_{ad2} V_{s1}(s)
\] (50)

With the eigenmodes \((\gamma_1, \gamma_2)\) for the propagation eigenvalue are

\[
\gamma_1 = s\sqrt{(C_s + C_m)(l_1 - l_m)} \quad \text{and} \quad \gamma_2 = s\sqrt{(C_s - C_m)(l_1 + l_m)}.
\] (52)

Comparing to the truly single line model (31,32), (50,51) can be regarded as the single line model of two coupled lines. Hence, the reflection coefficients in (43,44) are obtained.
source impedance of domain equations are only the first order terms are selected, the equivalent time only one reflection is considered. After the Taylor expansion, reflections are considered. A very specialized case is that if simulation and modeling [14], only a limited number of include infinite number of reflections. It will take computer a

Similarly, the closed forms for the far end are

\[ V_1(l,s) = m_{11} \cdot v_{dd1}(l,s) + m_{12} \cdot v_{dd2}(l,s) \]

\[ = \frac{1}{2} \left( \frac{1}{1 - \rho_{ld1} e^{-2i\gamma_1 l}} \cdot \tau_{ad1} \right) \]

\[ V_2(l,s) = m_{21} \cdot v_{dd1}(l,s) + m_{22} \cdot v_{dd2}(l,s) \]

\[ = \frac{1}{2} \left( \frac{1}{1 - \rho_{ld1} e^{-2i\gamma_1 l}} \cdot \tau_{ad1} \right) \]

Since (50,51) and (57,58) are the closed forms, they include infinite number of reflections. It will take computer a long time to converge. In order to reduce the complexity in simulation and modeling [14], only a limited number of reflections are considered. A very specialized case is that if only one reflection is considered. After the Taylor expansion, only the first order terms are selected, the equivalent time domain equations are

\[ v_1(0,t) = \frac{1}{2} \left[ \tau_{ad1} v_{sl}(t) + \tau_{ad2} v_{sl}(t) \right] \]

\[ + \tau_{ad1} \rho_{ld1}(1 + \rho_{ld1}) v_{sl}(t - 2Td_1) \]

\[ + \tau_{ad2} \rho_{ld2}(1 + \rho_{ld2}) v_{sl}(t - 2Td_2) \],

\[ v_2(0,t) = \frac{1}{2} \left[ -\tau_{ad1} v_{sl}(t) + \tau_{ad2} v_{sl}(t) \right] \]

\[ - \tau_{ad1} \rho_{ld1}(1 + \rho_{ld1}) v_{sl}(t - 2Td_1) \]

\[ + \tau_{ad2} \rho_{ld2}(1 + \rho_{ld2}) v_{sl}(t - 2Td_2) \].

Here, \( Td_i \) is the propagation delay of line \( i \).

\( Td_1 = \sqrt{C_1 + C_n} l_1 - l\_m, \quad Td_2 = \sqrt{C_1 - C_n} l_1 + l\_m. \)

(59, 60) are the near end waveforms for the aggressor and victim lines. They are composed of four terms which are (1) the even mode incident wave, (2) the odd mode incident wave, (3) the even mode reflected wave, and (4) the odd mode reflected waves. (61, 62) are the far end waveforms. Their 1\textsuperscript{st} terms are the even mode incident waves and 2\textsuperscript{nd} terms are the odd mode incident waves. To simplify the modeling and speed up the simulation, these terms are modeled as high pass or low pass filters [14] in order to be incorporated into Matlab. From each terms in (59-61), we are able to check the effects of every individual coupling effects.

5. N-coupled transmission lines

A N-coupled transmission line system can be modeled as an equivalent single line system by the similar derivation in the last section based on the same assumptions. When considering only one source \( V_{sl}(s) \), the closed forms for \( j \)th line is

\[ V_i(0,s) = \sum_{j=1}^{N} m_{ij} m_{lj} V_{ddj}(0,s), \quad \text{for} \quad i = 1, ..., N \]

\[ V_i(l,s) = \sum_{j=1}^{N} m_{ij} m_{lj} V_{ddj}(l,s), \quad \text{for} \quad i = 1, ..., N. \]

Here, \( V_{ddj}(0,s) \) and \( V_{ddj}(l,s) \) are the waveforms at both ends of the equivalent line \( j \) as illustrated in Figure 3. Their closed forms are

\[ V_{ddj}(0,s) = \frac{1 + \rho_{ld} e^{-2Td_{j}}}{1 - \rho_{ld} e^{-2Td_{j}}} \cdot \tau_{adj} V_{sl}(s), \]

\[ V_{ddj}(l,s) = \frac{1 + \rho_{ld} e^{-2Td_{j}}}{1 - \rho_{ld} e^{-2Td_{j}}} \cdot \tau_{adj} V_{sl}(s). \]

\[ Td_{j} = l_{j} \cdot \left[ C_{1} + 2C_m \cos\left(\frac{j\pi}{N+1}\right)\right], \quad \left[ C_{1} - 2C_m \cos\left(\frac{j\pi}{N+1}\right)\right], \]

for \( j = 1, ..., N. \)

The equivalent transmission and reflection coefficients are derived as
\[ \tau_{adj} = \frac{z_{adj}}{x_{Sj} + z_{adj}}, \quad \text{for } j = 1, \ldots, N. \]  
\[ \rho_{adj} = \frac{x_{Sj} - z_{adj}}{x_{Sj} + z_{adj}}, \quad \text{for } j = 1, \ldots, N. \]  
\[ \rho_{Ladj} = \frac{x_{Lj} - z_{adj}}{x_{Lj} + z_{adj}}, \quad \text{for } j = 1, \ldots, N. \]

Similar to (56), the closed forms for the equivalent characteristic impedances are

\[ Z_{adj} = \left[ l_i - 2n_m \cos\left(\frac{j\pi}{N+1}\right) \right] x_i + 2x_m \cos\left(\frac{j\pi}{N+1}\right), \quad \text{for } j = 1, \ldots, N. \]  

After Taylor expansion in time domain, each reflection terms can be derived from (33,34). The \( n \)th reflections for \( n \)th equivalent line are

\[ v_{dd}^{(n)}(0,t) = \rho_{Sdj}^{n-1} \rho_{Ldj}^{n-1} (1 + \rho_{Sdj}) \cdot \tau_{adj} v_{st}(t - 2nT_d), \quad \text{for } i = 1, \ldots, N. \]  
\[ v_{dd}^{(n)}(l,t) = \rho_{Sdj}^{n-1} \rho_{Ldj}^{n-1} (1 + \rho_{Sdj}) \cdot \tau_{adj} v_{st}(t - (2n-1)T_d), \quad \text{for } i = 1, \ldots, N. \]

Hence, the \( n \)th order closed forms can be given

\[ v_i(0,t) = \sum_{j=1}^{N} m_{ij} m_{ij} \sum_{k=1}^{n} v_{dd}^{(k)}(0,t), \quad \text{for } i = 1, \ldots, N. \]  
\[ v_i(l,t) = \sum_{j=1}^{N} m_{ij} m_{ij} \sum_{k=1}^{n} v_{dd}^{(k)}(l,t), \quad \text{for } i = 1, \ldots, N. \]

(64, 65), a S-domain closed form, represent infinite number of reflections in time domain. However, in (75,76), limited number of terms are chosen to minimize the modeling and simulation complexity. Hence, accuracy analysis is needed to determine the number of terms to be considered.

### 6. Accuracy Analysis

Considering \( n \)th order reflection for \( N \)-coupled lines, the qualitative error ratio at near end is derived as follows,

\[ \epsilon_{near,i} = \frac{n m_{ij} m_{ij} (1 + \rho_{Sdj}) \rho_{Ldj}^{n-1} \rho_{Ldj}^{n-1}}{1 + \rho_{Ldj}}, \quad \text{for } i = 1, \ldots, N. \]  

Similarly, error ratio at far end is

\[ \epsilon_{far,i} = \frac{n m_{ij} m_{ij} \rho_{Sdj}^{n-1} \rho_{Ldj}^{n-1}}{1 + \rho_{Ldj}}. \quad \text{for } i = 1, \ldots, N. \]  

After the modeling and error analysis, the experimental results are presented next.

### 7. Experimental Results

In order to illustrate the properties of the single-line equivalent modeling, we use test circuits with two and three coupled lines. The first case of two coupled lines is shown in Figure 4. The length of the lines is 3 inches and their

Figure 4. Two coupled transmission lines case.

| Table 1. Error Analysis for two coupled line case. |
|---|---|---|---|---|---|
| Error | \( \epsilon_{near,1} \) | \( \epsilon_{near,2} \) | \( \epsilon_{far,1} \) | \( \epsilon_{far,2} \) |
| n=1 | \( 1.14 \times 10^{-3} \) | \( 2.80 \times 10^{-3} \) | \( 6.09 \times 10^{-7} \) | \( 1.11 \times 10^{-2} \) |
| n=2 | \( 7.23 \times 10^{-3} \) | \( 2.95 \times 10^{-3} \) | \( 3.80 \times 10^{-5} \) | \( 1.60 \times 10^{-1} \) |

Figure 5. Three coupled transmission lines case.

| Table 2. Error Analysis for three coupled line case |
|---|---|---|---|---|---|
| Near End | Far End |
| \( \epsilon_{near,1} \) | \( \epsilon_{near,2} \) | \( \epsilon_{near,3} \) | \( \epsilon_{far,1} \) | \( \epsilon_{far,2} \) | \( \epsilon_{far,3} \) |
| n=1 | \( 1.56 \times 10^{-7} \) | \( 1.06 \times 10^{-7} \) | \( 7.01 \times 10^{-7} \) | \( 5.88 \times 10^{-7} \) | \( 8.1 \times 10^{-3} \) | \( 2.08 \times 10^{-4} \) |
| n=2 | \( 9.72 \times 10^{-1} \) | \( 4.22 \times 10^{-1} \) | \( 6.24 \times 10^{-2} \) | \( 3.8 \times 10^{-3} \) | \( 1.4 \times 10^{-2} \) | \( 1.41 \times 10^{-4} \) |
electrical parameters are given in Figure 4. They are tri-diagonal and symmetric Toeplitz matrices. The mismatch terminations at both ends are $R_s = 30\Omega$, $R_a = 30\Omega$, and $C = 5p$. The input ramp signal is applied to line 1 with an amplitude of 1.7V and a rise time of 350ps.

From the error analysis in Section 5, the steady state errors for the first and second order approximation are shown in Table 1. The error is 6.09% for $n=1$ and 0.38% for $n=2$. In other words, when considering only two reflections, the error is very small and neglectable.

Figure 6 shows the simulations results considering only the second order reflection. The first stripe is the near end waveform of the aggressor line. It is the summation of the next three stripes. They are the incident waves in stripe 2, the 1st order reflections in stripe 3, and the 2nd order reflections in stripe 4. Note that there are odd and even mode reflections in each stripe. The remaining stripes are for the near end of the victim line, the far end of the aggressor, and the far end of the victim. As one can see, the 5th and 12th stripes are the crosstalk noise at near and far ends of the victim line. Each has two lines in it, one by HSPICE simulation and another by our qualitative analysis. The differences between them are in accord with the error analysis results shown in Table 1.

The second case with three coupled lines is shown in Figure 5. Only the first order reflection is considered. The error analysis is shown in Table 2. The maximal error is 5.88% at the far end of the first line. Although the error is 7.01% for the near end of the third line, it amplitude is very small as compared to the first line. Figure 7 shows the simulation results. They are arranged in the same manner as Figure 6. Stripe 1 and 10 show the waveforms at near and far ends of the line 1. Stripe (4,12) and (7, 14) show the waveforms for line 2 and 3. In addition to the overall waveforms in these stripes, the remaining stripes show different orders of the crosstalk.

Conclusions

In this paper, we have proposed a methodology to model multiple coupled transmission lines as a single uncoupled transmission line to minimize the simulation complexity. In additional to the quantitative analysis of the overall waveform, it also produces the qualitative analysis of each individual reflection terms. These waveforms provide valuable information for the design and diagnosis of the printed circuit boards. Modeling methodology and error analysis are conducted mathematically and implemented in MATLAB. The thorough experiments on two and three coupled line systems show that the proposed models are able to achieve the accuracy within the tolerance derived by the error analysis.

References


Figure 6. Simulation results for two coupled transmission lines with the second order reflections.

Figure 7. Simulation results for three coupled transmission lines with the first order reflection.