The Spanning Diameter of the Star Graphs

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Abstract

Assume that $u$ and $v$ are any two distinct vertices of different partite sets of the
$n$-dimensional star graph $S_n$ with $n \geq 5$. We prove that there are $(n-1)$ internally
disjoint paths $P_1, P_2, \ldots, P_{n-1}$ joining $u$ to $v$ such that $\bigcup_{i=1}^{n-1} P_i$ spans $S_n$ and $l(P_i) \leq (n-1)! + 2(n-2)! + 2(n-3)! + 1 = \frac{n!}{n-2} + 1$. We also prove that there are two internally
disjoint paths $Q_1$ and $Q_2$ joining $u$ to $v$ such that $Q_1 \cup Q_2$ spans $S_n$ and $l(Q_i) \leq \frac{n!}{2} + 1$ for $i = 1, 2$.

Keywords: diameter, hamiltonian, hamiltonian laceable, star graphs.

1 Basic Definitions

For the graph definition and notation we follow [3]. $G = (V, E)$ is a graph if $V$ is a finite set
and $E$ is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that $V$ is the vertex
set and $E$ is the edge set. For a vertex $u$, $N(u)$ denotes the neighborhood of $u$ which is the set
$\{v \mid (u, v) \in E\}$. For any vertex $x$ of $V$, $\deg_G(x)$ denotes its degree in $G$. We use $\delta(G)$
to denote $\min\{\deg_G(x) \mid x \in V\}$, and we use $\Delta(G)$ to denote $\max\{\deg_G(x) \mid x \in V\}$. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E$. A path is represented by $\langle v_0, v_1, \ldots, v_k \rangle$. The length of a path $Q$, $l(Q)$, is the number of edges in $Q$. We also write the path $\langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_l, \ldots, v_k \rangle$, where $Q_1$ is the path $\langle v_0, v_1, \ldots, v_i \rangle$ and $Q_2$ is the path $\langle v_j, v_{j+1}, \ldots, v_l \rangle$. Hence, it is possible to write a path as $\langle v_0, v_1, Q, v_1, v_2, \ldots, v_k \rangle$ if $l(Q) = 0$. 

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We use \(d(u, v)\) to denote the distance between \(u\) and \(v\), i.e., the length of the shortest path joining \(u\) and \(v\). The diameter of a graph \(G\), \(D(G)\), is defined as \(\max\{d(u, v) \mid u, v \in V\}\). A path is a hamiltonian path if it contains all vertices of \(G\). A graph \(G\) is hamiltonian connected if there exists a hamiltonian path joining any two distinct vertices of \(G\). A cycle is a path with at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of \(G\) is a cycle that traverses every vertex of \(G\) exactly once. A graph is hamiltonian if it has a hamiltonian cycle.

The connectivity of \(G\), \(\kappa(G)\), is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger’s Theorem [11] that there are \(k\) internally vertex-disjoint (abbreviated as disjoint) paths joining any two distinct vertices \(u\) and \(v\) when \(k \leq \kappa(G)\). A container \(C(u, v)\) between two distinct vertices \(u\) and \(v\) in \(G\) is a set of internally disjoint paths \(\{P_1, P_2, \ldots, P_k\}\) between \(u\) and \(v\). The width of \(C(u, v)\) is \(k\). A \(w\)-container is a container of width \(w\). The length of a \(C(u, v) = \{P_1, \ldots, P_k\}\), \(l(C(u, v))\), is \(\max\{l(P_i) \mid 1 \leq i \leq k\}\). The \(w\)-wide distance between \(u\) and \(v\), \(d_w(u, v)\), is \(\min\{l(C(u, v)) \mid C(u, v)\text{ is a }w\text{-container}\}\). The \(w\)-diameter of \(G\), \(D_w(G)\), is \(\max\{d_w(u, v) \mid u, v \in V, u \neq v\}\). In particular, the wide diameter of \(G\) is \(D_{\kappa(G)}(G)\). The wide diameter is an important measure for interconnection networks [7].

In this paper, we are interested in a specific type of containers. We say that a \(w\)-container \(C(u, v)\) is a \(w^\ast\)-container if every vertex of \(G\) is incident with a path in \(C(u, v)\). A graph \(G\) is \(w^\ast\)-connected if there exists a \(w^\ast\)-container between any two distinct vertices \(u\) and \(v\). Obviously, a graph \(G\) is \(1^\ast\)-connected if and only if it is hamiltonian connected. Moreover, a graph \(G\) is \(2^\ast\)-connected if it is hamiltonian. The study of \(w^\ast\)-connected graph is motivated by the globally \(3^\ast\)-connected graphs proposed by Albert, Aldred, and Holton [2]. A globally \(3^\ast\)-connected graph is a 3-regular \(3^\ast\)-connected graph. Assume that a graph \(G\) is \(w^\ast\)-connected. Obviously, \(w \leq \kappa(G) \leq \delta(G) \leq \Delta(G)\). A graph \(G\) is super spanning connected if \(G\) is \(w^\ast\)-connected for any \(w\) with \(1 \leq w \leq \kappa(G)\). In [10], it is proved that the pancake graph \(P_n\) is super spanning connected if and only if \(n \neq 3\).

A graph \(G\) is bipartite if its vertex set can be partitioned into two subsets \(V_1\) and \(V_2\) such that every edge joins vertices between \(V_1\) and \(V_2\). Let \(G\) be a \(k\)-connected bipartite graph with bipartition \(V_1\) and \(V_2\) such that \(|V_1| \geq |V_2|\). Suppose that there exists a \(k^\ast\)-container \(C(u, v) = \{P_1, P_2, \ldots, P_k\}\) in a bipartite graph \(G\) joining \(u\) to \(v\) with \(u, v \in V_1\). Since the length of \(P_i = 2k_i + 1\) is an odd integer for all \(1 \leq i \leq k\), there are \(k_i - 1\) vertices of \(P_i\) in \(V_1\) other than \(u\) and \(v\), and \(k_i\) vertices of \(P_i\) in \(V_2\). As a consequence, \(|V_1| = \sum_{i=1}^{k}(k_i - 1) + 2\) and \(|V_2| = \sum_{i=1}^{k} k_i\). Therefore, any bipartite graph \(G\) with \(\kappa(G) \geq 3\) is not \(k^\ast\)-connected. For this reason, we define that a bipartite graph is \(k^\ast\)-laceable if there exists a \(k^\ast\)-container between any two vertices from different partite sets. Obviously, any \(k^\ast\)-laceable graph with \(k \geq 1\) is a bipartite graph with the equal size of bipartition. An \(1^\ast\)-laceable graph is also known as hamiltonian laceable graph. Moreover, a bipartite graph is \(2^\ast\)-laceable if and only if it is hamiltonian. Obviously, all \(1^\ast\)-laceable graphs except \(K_1\) and \(K_2\) are \(2^\ast\)-laceable. A \(k\)-regular bipartite graph \(G\) is super spanning laceable if \(G\) is \(i^\ast\)-laceable for all \(1 \leq i \leq \kappa(G)\). It was proved in [10] that the star graph \(S_n\) is super spanning laceable if and only if \(n \neq 3\).
We also define the \( w^* \)-lacable distance between any two vertices \( u \) and \( v \) from different partite sets, \( d_w^{s^*}(u, v) \), is \( \min\{l(C(u, v)) \mid C(u, v) \text{ is a } w^*\text{-container}\} \). The \( w^* \)-diameter of \( G \), denoted by \( D_w^{s^*}(G) \), is \( \max\{d_w^{s^*}(u, v) \mid u \text{ and } v \text{ are vertices from different partite sets}\} \).

In particular, the wide spanning diameter of \( G \) is \( D_w^{s^*}(G) \). In this paper, we prove that \( D_w^{s^*}(S_n) = (n - 1)! + 2(n - 2)! + 2(n - 2)! + 1 = \frac{n!}{n - 2} + 1 \) and \( D_w^{s^*}(n) = \frac{n!}{2} + 1 \).

In Section 2, we give the definition of the star graphs and introduce some basic properties of star graphs. Then we prove that \( D_w^{s^*}(S_n) = (n - 1)! + 2(n - 2)! + 2(n - 2)! + 1 = \frac{n!}{n - 2} + 1 \) in Section 3. In Section 4, we prove that \( D_w^{s^*}(S_n) = \frac{n!}{2} + 1 \). We conclude the paper in the final section.

## 2 The star graphs and some notation conventions

Let \( n \) be a positive integer. We use \( \langle n \rangle \) to denote the set \( \{1, \ldots, n\} \). The \( n \)-dimensional star graph, denoted by \( S_n \), is a graph with the vertex set \( V(S_n) = \{u_1 \ldots u_n \mid u_i \in \langle n \rangle \} \) and \( u_i \neq u_j \) for \( i \neq j \). The adjacent is defined as follows: \( u_1 \ldots u_i \ldots u_n \) is adjacency to \( v_1 \ldots v_i \ldots v_n \) through an edge of dimension \( i \) with \( 2 \leq i \leq n \) if \( v_j = u_j \) for \( j \notin \{1, i\} \), \( v_1 = u_i \), and \( v_i = u_1 \). The star graphs \( S_2, S_3, \) and \( S_4 \) are illustration in Figure 1. The star graphs are an important family of interconnection networks proposed by Akers and Krishnameurthy [1]. Some interesting properties of star graphs are studied [4, 5, 8, 12, 13]. It is known that the connectivity of \( S_n \) is \( n - 1 \). We use bold face to denote vertices in \( S_n \). Hence \( u_1, u_2, \ldots, u_n \) is denotes a sequence of vertices in \( S_n \).

![Figure 1: The star graphs S2, S3, and S4.](image)

By definition, \( S_n \) is an \( (n - 1) \)-regular graph with \( n! \) vertices. Moreover, it is vertex transitive and edge transitive. We use \( e \) to denote the element \( 12 \ldots n \). It is known that \( S_n \)
is a bipartite graph with one partite set containing those vertices corresponds to odd permutations and the other partite set containing those vertices corresponds to even permutations. We will use white vertices to represent those even permutation vertices and use black vertices to represent those odd permutation vertices. Let \( u = u_1u_2 \ldots u_n \) be any vertex of the star graph \( S_n \). We say that \( u_i \) is the \( i \)-th coordinate of \( u \), denoted by \((u)_i\), for \( 1 \leq i \leq n \). By the definition of \( S_n \), there is exactly one neighbor \( v \) of \( u \) such that \( u \) and \( v \) are adjacent through an \( i \)-dimensional edge with \( 2 \leq i \leq n \). For this reason, we use \((u)^i\) to denote the unique \( i \)-neighbor of \( u \). Obviously, \((u)^i = u \). For \( 1 \leq i \leq n \), let \( S_n^{(i)} \) denote the subgraph of \( S_n \) induced by those vertices \( u \) with \((u)_n = i \). Obviously, \( S_n \) can be decomposed into \( n \) subgraph \( S_n^{(i)} \), \( 1 \leq i \leq n \), and each \( S_n^{(i)} \) is isomorphic to \( S_{n-1} \). Thus, the star graph can be constructed recursively. Let \( I \subseteq \langle n \rangle \). We use \( S_I \) to denote the subgraph of \( S_n \) induced by \( \bigcup_{i \in I} V(S_n^{(i)}) \). For \( 1 \leq i \neq j \leq n \), we use \( E^{i,j} \) to denote the set of edges between \( S_n^{(i)} \) and \( S_n^{(j)} \).

For \( 1 \leq i \neq j \leq n \), we use \( S_n^{(i,j)} \) to denote the subgraph of \( S_n \) induced by those vertices \( u \) with \((u)_{n-1} = i \) and \((u)_n = j \). Obviously, \( S_n^{(i,j)} \neq S_n^{(j,i)} \) and \( S_n^{(i,j)} \) is isomorphic \( S_{n-2} \). Obviously, we have the following lemmas.

**Lemma 1** Let \( u = u_1u_2 \ldots u_n \). Then \( u \) is a white vertex if and only if \( (u)_n = (n-2)! \) for any \( 1 \leq i \neq j \leq n \). Moreover, there are \((n-2)!/2\) edges joining black vertices of \( S_n^{(i)} \) to white vertices of \( S_n^{(j)} \).

**Lemma 2** Assume that \( u \) and \( v \) are any two distinct vertices of \( S_n \) with \( 1 \leq d(u, v) \leq 2 \). Then \((u)_1 \neq (v)_1 \).

**Theorem 1** [8] \( S_n \) is hamiltonian laceable if and only if \( n \neq 3 \).

**Theorem 2** [8] For any black vertex \( w \) and any two distinct white vertices \( u, v \) of \( S_n \) with \( n \geq 4 \). There exists a hamiltonian path of \( S_n \) joining \( u \) to \( v \).

**Lemma 3** Assume that \( n \geq 5 \) and \( I = \{i_1, i_2, \ldots, i_m\} \) is a nonempty subset of \( \langle n \rangle \). Let \( u \) be a white vertex and \( v \) be a black vertex of \( S_n^I \) with \((u)_{i_1} = i_1 \) and \((v)_{i_m} = i_m \). Then there exists a hamiltonian path of \( S_n^I \) joining \( u \) to \( v \).

**Proof.** Obviously, \( S_n^{(i,j)} \) is isomorphic to \( S_{n-1} \) for all \( 1 \leq j \leq m \). By Theorem 1, this lemma is true on \( m = 1 \). Thus, we assume that \( m \geq 2 \). We set that \( x_1 = u \) and set that \( y_m = v \). By Lemma 1, \(|E^{i,j} = (n-2)!|\). We choose \((y_j, x_{j+1}) \in E^{i,j+1}\) with \( y_j \) is a black vertex and \( x_{j+1} \) is a white vertex for \( 1 \leq j \leq m - 1 \). By Theorem 1, there is a hamiltonian path \( P_j \) of \( S_n^{(i,j)} \) joining \( x_j \) to \( y_j \) for all \( 1 \leq j \leq m \). Then \( (x_1, P_1, y_1, x_2, P_2, y_2, \ldots, x_m, P_m, y_m) \) forms a hamiltonian path of \( S_n^I \) joining \( u \) to \( v \).

**Lemma 4** Assume that \( r \) and \( s \) are any two adjacent vertices of \( S_n \) with \( n \geq 4 \). Then there exists a hamiltonian path of \( S_n \) joining any white vertex \( u \) to a black vertex \( v \) with \((v)_1 = i \) for any \( i \in \langle n \rangle \).

**Proof.** Since \( S_n \) is vertex transitive and edge transitive, we assume that \( r = e \) and \( s = (e)^2 \). Obviously, \( r, s \in S_n^{(n)} \). We prove this lemma by induction on \( n \). The required hamiltonian paths of \( S_4 \) are listed below.
Theorem 3

There exists a hamiltonian path $S$ and $(S_1)$ the theorem is true on either $v$ is a black vertex of $S$.

Case 1: $u \in S^{(n)}$. Obviously, $S^{(n)}$ is isomorphism to $S^{n-1}$. By induction, there exists a hamiltonian path $P$ of $S^{(n)} - \{r, s\}$ joining $u$ to a black vertex $x$ with $(x)_1 = n - 1$. Let $v$ is a black vertex of $S^{(n-2)}$ with $(v)_1 = i$. By Lemma 3, there exists a hamiltonian path $Q$ of $S^{(n-1)}$ joining the white vertex $(x)_n$ to $v$. Then $(u, P, x, (x)_n, Q, v)$ forms the desired path.

Case 2: $u \in S^{(k)}$ for some $k \neq n$. By Lemma 1, there exists $(n - 2)!/2 \geq 3$ edges joining black vertices of $S^{(k)}$ to white vertices of $S^{(n)}$. We can choose a black vertex $x$ of $S^{(k)}$ such that $(x)_n$ is a white vertex of $S^{(n)} - \{r, s\}$. Since $S^{(k)}$ is isomorphism to $S^{n-1}$, by Theorem 1, there exists a hamiltonian path $P$ of $S^{(k)}$ joining $u$ to $x$. By induction, there exists a hamiltonian path $Q$ of $S^{(n)} - \{r, s\}$ joining $(x)_n$ to a black vertex $y$ with $(y)_1 = r \in \langle n - 1 \rangle - \{k\}$. Let $v$ is a black vertex of $S^{(n-1)-(k,r)}$ with $(v)_1 = i$. By Lemma 3, there exists a hamiltonian path $R$ of $S^{(n-1)-(k)}$ joining the white vertex $(y)_n$ to $v$. Obviously, $(u, P, x, (x)_n, Q, y, (y)_n, R, v)$ forms the desired path.

Theorem 3 Assume that $n \geq 5$ and $I = \{i_1, i_2, \ldots, i_m\}$ is a nonempty subset of $\langle n \rangle$. Then $S^I_n$ is hamiltonian laceable.

Proof. Suppose that $u$ is a white vertex and $v$ is a black vertex of $S^I_n$. By Lemma 3, this theorem is true on either $m = 1$, or $m \geq 2$ and $(u)_n \neq (v)_n$. Thus, we assume that $m \geq 2$ and $(u)_n = (v)_n$. Without loss of generality, we assume that $(u)_n = (v)_n = i_1$. 

Assume that this lemma is hold on $S_k$ for all $4 \leq k < n$. We have the following cases:

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Case 1: \((v)_1 \in I\). Without loss of generality, we assume that \((v)_1 = i_m\). Obviously, \(S_n^{(i_1)}\) is isomorphic to \(S_{n-1}\). By Theorem 2, there exists a hamiltonian path \(P\) of \(S_n^{(i_1)} - \{v\}\) joining \(u\) to a white vertex \(x\) such that \((x)_1 = i_2\). By Lemma 3, there exists a hamiltonian path \(Q\) of \(S_n^{(i_1-i)}\) joining the black vertex \((x)_n\) to the white vertex \((v)_n\). Then the path \(<u, P, x, (x)_n, Q, (v)_n, v>\) forms a hamiltonian path of \(S_n^I\) joining \(u\) to \(v\).

Case 2: \((u)_1 \notin I\) and \((v)_1 \notin I\). We can choose a white vertex \(y\) be a neighbor of \(v\) in \(S_n^{(i_1)}\) with \((y)_n = i_m\). Obviously, \(y \neq u\). By Lemma 4, there exists a hamiltonian path \(P\) of \(S_n^{(i_1)} - \{v, y\}\) joining \(u\) to a black vertex \(x\) such that \((x)_n = i_2\). By Lemma 3, there exists a hamiltonian path \(Q\) of \(S_n^{(i_1-i)}\) joining the white vertex \((x)_n\) to the black vertex \((y)_n\). Then the path \(<u, P, x, (x)_n, Q, (y)_n, y, v>\) forms a hamiltonian path of \(S_n^I\) joining \(u\) to \(v\).

\[\square\]

**Theorem 4** Assume that \(n \geq 5\). \(S_n - \{r, s\}\) is hamiltonian laceable for any adjacent vertices \(r\) and \(s\).

**Proof.** Since the \(S_n\) is vertex transitive and edge transitive, we assume that \(r = e\) and \(s = (e)^2\). Obviously, \(r, s \in S_{n}^{(n)}\). Suppose that \(u\) is a white vertex and \(v\) is a black vertex of \(S_n - \{r, s\}\). We want to find a hamiltonian path of \(S_n - \{r, s\}\) joining \(u\) to \(v\).

Case 1: \(u, v \in S_{n}^{(n)}\). By Lemma 4, there exists a hamiltonian path \(P\) of \(S_{n}^{(n)} - \{r, s\}\) joining \(u\) to a black vertex \(y\) with \((y)_1 \in \langle n - 1 \rangle\). Obviously, \(P\) can be write as \(<u, Q_1, x, v, Q_2, y>\). (Note that \(l(Q_1) = 0\) if \(u = x\) and \(l(Q_2) = 0\) if \(v = y\).) By Theorem 3, there exists a hamiltonian path \(R\) of \(S_{n}^{(n-1)}\) joining the black vertex \((x)_n\) to the white vertex \((y)_n\). Then \(<u, Q_1, x, (x)_n, R, (y)_n, y, (Q_2)^{-1}, v>\) forms a desired hamiltonian path.

Case 2: \(u, v \in S_{n}^{(k)}\) for some \(k \neq n\). By Theorem 1, there exists a hamiltonian path \(P\) of \(S_{n}^{(k)}\) joining \(u\) to \(v\). By Lemma 1, there exists \((n - 2)!/2 \geq 3\) edges joining black vertices of \(S_{n}^{(k)}\) to white vertices of \(S_{n}^{(n)}\). We can choose a black vertex \(x\) of \(S_{n}^{(k)}\) with \((x)_1 = n\) and \((x)_n \notin \{r, s\}\). Obviously, \(P\) can be write as \(<u, Q_1, x, y, Q_2, v>\). (Note that \(l(Q_1) = 0\) if \(u = x\) and \(l(Q_2) = 0\) if \(v = y\).) Obviously, \(d(x, y) = 1\). By Lemma 2, \((x)_1 \neq (y)_1\). By Lemma 4, there exists a hamiltonian path \(R\) of \(S_{n}^{(n)} - \{r, s\}\) joining the white vertex \((x)_n\) to a black vertex \(z\) with \((z)_1 \in \langle n - 1 \rangle - \{k\}\). By Theorem 3, there exists a hamiltonian path \(T\) of \(S_{n}^{(n-1)}\) joining the black vertex \((z)_n\) to the white vertex \((y)_n\). Obviously, \(<u, Q_1, x, (x)_n, R, z, (z)_n, T, (y)_n, y, Q_2, v>\) forms a desired hamiltonian path.

Case 3: \(u \in S_{n}^{(n)}\) and \(v \in S_{n}^{(k)}\) for some \(k \neq n\). By Lemma 4, there exists a hamiltonian path \(P\) of \(S_{n}^{(n)} - \{r, s\}\) joining \(u\) to a black vertex \(x\) with \((x)_1 \in \langle n - 1 \rangle\). By Theorem 3, there exists a hamiltonian path \(Q\) of \(S_{n}^{(n-1)}\) joining the white vertex \((x)_n\) to \(v\). Then \(<u, P, x, (x)_n, Q, v>\) forms a desired hamiltonian path.

Case 4: \(u \in S_{n}^{(k)}\) and \(v \in S_{n}^{(l)}\) with \(k, l, \) and \(n\) are distinct. By Lemma 1, there exists \((n - 2)!/2 \geq 3\) edges joining black vertices of \(S_{n}^{(k)}\) to white vertices of \(S_{n}^{(n)}\), we can choose a black vertex \(x\) of \(S_{n}^{(k)}\) with \((x)_1 = n\) and \((x)_n \notin \{r, s\}\). By Theorem 1, there exists a hamiltonian path \(P\) of \(S_{n}^{(k)}\) joining \(u\) to \(x\). By Lemma 4, there exists a hamiltonian path \(Q\).
of $S_n^{\{n\}} - \{r, s\}$ joining the white vertex $(x)^n$ to a black vertex $y$ with $(y)_1 \in \langle n - 1 \rangle - \{k\}$. By Lemma 3, there exists a hamiltonian path $R$ of $S_n^{\{n-1\} - \{k\}}$ joining the white vertex $(y)^n$ to $v$. Then $(u, P, x, (x)^n, Q, y, (y)^n, R, v)$ forms a desired hamiltonian path. \hfill \Box

**Lemma 5** Assume that $n \geq 4$. Let $u$ be a white vertex of $S_n$ and $F_n = \{(u)^i \mid 3 \leq i \leq n\} \cup \{((u)^i)^{-1} \mid 3 \leq i \leq n\}$. Then there exists a hamiltonian path of $S_n - F_n$ joining $u$ to a black vertex $v$ with $(v)_1 = j$ for any $j \in \langle n \rangle$.

**Proof.** We proved this lemma by induction on $n$. Since $S_n$ is vertex transitive, we assume that $u = e$. The required hamiltonian paths of $S_4 - F_4$ are listed below.

Assume that the lemma is true for any $S_k$ with $4 \leq k \leq n - 1$. Obviously, $S_n^{\{n\}}$ is isomorphic to $S_{n-1}$. By induction, there exists a hamiltonian path $P$ of $S_n^{\{n\}} - F_n - 1$ joining $e$ to a black vertex $x$ with $(x)_1 = 1$. By Lemma 4, there exists a hamiltonian path $Q$ of $S_n^{\{1\}} - \{(e)^n, ((e)^n)^{-1}\}$ joining the white vertex $(x)^n$ to a black vertex $y$ with $(y)_1 = 2$. We can choose a black vertex $z$ of $S_n^{(n-1) - \{1\}}$ with $(z)_1 = i$. By Theorem 3, there exists a hamiltonian path $R$ of $S_n^{\{n\} - \{1, n\}}$ joining the white vertex $(y)^n$ to $z$. Then $(e, P, x, (x)^n, Q, y, (y)^n, R, z)$ forms a desired hamiltonian path. \hfill \Box

**Lemma 6** Assume that $n \geq 5$. Suppose that $p$ and $q$ are two different white vertices of $S_n$, and $r$ and $s$ are two different black vertices of $S_n$. Then there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ joins $p$ to $r$, (2) $P_2$ joins $q$ to $s$, and (3) $P_1 \cup P_2$ spans $S_n$.

**Proof.** Without loss of generality, we assume that $(p)_n = n$ and $(q)_n = n - 1$. Let $(r)_n = i$ and $(s)_n = j$.

**Case 1:** $i, j \in \langle n - 2 \rangle$ and $i \neq j$. By Theorem 3, there exists a hamiltonian path $P_1$ of $S_n^{(i, n)}$ joining $p$ to $r$. Again, there exists a hamiltonian path $P_2$ of $S_n^{(n-1) - \{i\}}$ joining $q$ to $s$. Then $P_1$ and $P_2$ form the desired paths.

**Case 2:** $i, j \in \langle n - 2 \rangle$ and $i = j$. We can choose a white vertex $x$ be the neighborhood of $r$ in $S_n^{(i)}$ with $(x)_1 \neq n - 1$. By Theorem 3, there exists a hamiltonian path $P$ of $S_n^{(n) - \{i, n-1\}}$ joining $p$ to the black vertex $(x)^n$. By Lemma 4, there exists a hamiltonian path $Q$ of $S_n^{(i) - \{r, x\}}$ joining $s$ to a black vertex $y$ with $(y)_1 = n - 1$. By Theorem 1, there exists a hamiltonian path $Q$ of $S_n^{(n-1)}$ joining $q$ to the black vertex $(y)^n$. Then $P_1 = (p, P, (x)^n, x, r)$ and $P_2 = (q, Q, (y)^n, y, Q^{-1}, s)$ form the desired paths.

**Case 3:** either $i \in \langle n - 2 \rangle$ and $j = n - 1$, or $i = n$ and $j \in \langle n - 2 \rangle$. By symmetric, we assume that $i \in \langle n - 2 \rangle$ and $j = n - 1$. By Theorem 3, there exists a hamiltonian path $P_1$ of $S_n^{(n) - \{n-1\}}$ joining $p$ to $r$. By Theorem 1, there exists a hamiltonian path $P_2$ of $S_n^{(n-1)}$ joining $q$ to $s$. Then $P_1$ and $P_2$ are the desired paths.
Case 4: either \( i = n - 1 \) and \( j \in \{ n - 2 \} \), or \( i \in \{ n - 2 \} \) and \( j = n \). By symmetric, we assume that \( i = n - 1 \) and \( j \in \{ n - 2 \} \). By Theorem 1, there exists a hamiltonian path \( R \) of \( S_n^{(n-1)} \) joining \( q \) to \( r \). We can choose a white vertex \( x \in R \) with \( (x)^n \in S_n^n \). We can write \( R \) as \( \langle q, R_1, y, x, R_2, r \rangle \). By Theorem 1, there exists a hamiltonian path \( W \) of \( S_n^n \) joining \( p \) to the black vertex \( (x)^n \). We set \( P_1 \) as \( \langle p, W, (x)^n, x, R_2, r \rangle \). Obviously, \( y \) is a black vertex and \( (y)_1 \in \{ n - 2 \} \). By Theorem 3, there exists a hamiltonian path \( Q \) of \( S_n^{n-2} \) joining the white vertex \( (y)^n \) to \( s \). We set \( P_2 \) as \( \langle q, R_1, y, (y)^n, Q, s \rangle \). Then \( P_1 \) and \( P_2 \) forms the desired paths.

Case 5: \( i = n \) and \( j = n - 1 \). By Theorem 1, there exists a hamiltonian path \( P_1 \) of \( S_n^n \) joining \( p \) to \( r \). By Theorem 3, there exists a hamiltonian path \( P_2 \) of \( S_n^{n-1} \) joining \( q \) to \( s \). Then \( P_1 \) and \( P_2 \) are the desired paths.

Case 6: \( i = n - 1 \) and \( j = n \). We can choose a white vertex \( x \in S_n^n - \{ s \} \) with \( (x)_1 = n - 1 \). By Theorem 1, there exists a hamiltonian path \( R \) of \( S_n^n \) joining \( p \) to \( s \). Again, there exists a hamiltonian path \( Q \) of \( S_n^{n-1} \) joining \( q \) to \( r \). We can write \( R \) as \( \langle p, R_1, x, y, R_2, s \rangle \), and write \( Q \) as \( \langle q, Q_1, w, (x)^n, Q_2, r \rangle \). Obviously, \( y \) is a black vertex and \( w \) is a white vertex. Since \( d(x, y) = 1 \), by Lemma 2, \( (y)_1 \neq (x)_1 = n - 1 \). Since \( d((x)^n, w) = 1 \), by Lemma 2, \( (w)_1 \neq (x)_1 = n \). By Theorem 3, there exists a hamiltonian path \( W \) of \( S_n^{n-1} \) joining the black vertex \( (w)^n \) to the white vertex \( (y)^n \). Then \( P_1 = \langle p, R_1, x, (x)^n, Q_2, r \rangle \) and \( P_2 = \langle q, Q_1, w, (w)^n, W, (y)^n, y, R_2 r \rangle \) form a desired paths.

Case 7: either \( i = j = n \) or \( i = j = n - 1 \). By symmetric, we assume that \( i = j = n \). By Theorem 1, there exists a hamiltonian path \( P \) of \( S_n^n \) joining \( p \) to \( s \). We can write \( P \) as \( \langle p, R_1, r, x, R_2, s \rangle \). We set \( P_1 \) as \( \langle p, R_1, r \rangle \). Obviously, \( x \) is a white vertex and \( (x)_1 \in \{ n - 1 \} \). By Theorem 3, there exists a hamiltonian path \( Q \) of \( S_n^{n-1} \) joining \( q \) to the black vertex \( (x)^n \). Then \( P_1 \) and \( P_2 \) = \( \langle q, Q, (x)^n, x, R_2, q \rangle \) are the desired paths. 

3 The \((n-1)^*L\)-diameter of \( S_n \)

Lemma 7 Let \( u = u_1u_2u_3u_4 \) be any white vertex of \( S_4 \). There exist three paths \( P_1, P_2, \) and \( P_3 \) such that (1) \( P_1 \) joins \( u \) to the black vertex \( u_2u_4u_1u_3 \) with \( l(P_1) = 7 \), (2) \( P_2 \) joins \( u \) to the white vertex \( u_3u_4u_1u_2 \) with \( l(P_2) = 8 \), (3) \( P_3 \) joins \( u \) to the white vertex \( u_4u_1u_3u_2 \) with \( l(P_3) = 8 \), and (4) \( P_1 \cup P_2 \cup P_3 \) spans \( S_4 \).

Proof. Since \( S_4 \) is vertex transitive, we assume that \( u = 1234 \). Then we set

\[
\begin{align*}
P_1 &= \langle 1234, 3214, 4213, 1243, 2143, 4123, 1423, 2413 \rangle, \\
P_2 &= \langle 1234, 4231, 3241, 2341, 4321, 3421, 2431, 1432, 3412 \rangle, \text{ and} \\
P_3 &= \langle 1234, 2134, 3124, 1324, 2314, 4312, 1342, 3142, 4132 \rangle.
\end{align*}
\]

Obviously, \( P_1, P_2, \) and \( P_3 \) forms the desired paths. 

\[\square\]
Lemma 8 Let $u = u_1 u_2 u_3 u_4$ be any white vertex of $S_4$. Let $i_1 i_2 i_3$ be a permutation of $u_2, u_3$, and $u_4$. There exist four paths $P_1, P_2, P_3,$ and $P_4$ of $S_4$ such that (1) $P_1$ joins $u$ to a white vertex $w$ with $(w)_1 = i_1$ and $l(P_1) = 2$, (2) $P_2$ joins $u$ to a white vertex $x$ with $(x)_1 = i_2$ and $l(P_2) = 2$, (3) $P_3$ joins $u$ to a black vertex $y$ with $(y)_1 = i_3$ and $l(P_3) = 19$, (4) $P_4$ joins $u$ to a black vertex $z$ with $z \neq y$, $(z)_1 = i_3$, and $l(P_4) = 19$, (5) $P_1 \cup P_2 \cup P_3$ spans $S_4$, and (6) $P_1 \cup P_2 \cup P_4$ spans $S_4$.

Proof. Since $S_4$ is vertex transitive, we assume that $u = 1234$. Without loss of generality, we suppose that $i_1 < i_2$. The required four paths are listed below.

<table>
<thead>
<tr>
<th>Path $P_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 = (1234)(2134)(3124)$</td>
</tr>
<tr>
<td>$P_2 = (1234)(2143)(1243)(1234)$</td>
</tr>
<tr>
<td>$P_3 = (1234)(2341)(1342)(1234)$</td>
</tr>
<tr>
<td>$P_4 = (1234)(2431)(1432)(1234)$</td>
</tr>
</tbody>
</table>

Lemma 9 Assume that $n \geq 5$ and $i_1 i_2 \ldots i_{n-1}$ is an $(n-1)$-permutation of $(n)$. Let $u$ be any white vertex of $S_n$. Then there exist $(n-1)$ paths $P_1, P_2, \ldots, P_{n-1}$ of $S_n$ such that (1) $P_1$ is joins $u$ to a black vertices $y_1$ with $(y_1)_1 = i_1$ and $l(P_1) = n(n-2)! - 1$, (2) $P_j$ is joins $u$ to a white vertices $y_j$ with $(y_j)_1 = i_j$ and $l(P_j) = n(n-2)!$ for $2 \leq j \leq n-1$, and (3) $\cup_{j=1}^{n-1} P_j$ spans $S_n$.

Proof. Since $S_n$ is vertex transitive, we assume that $u = e$. Without loss of generality, we suppose that $i_2 < i_3 < \ldots < i_{n-1}$.

Case 1: $n = 5$. Obviously, $i_2 \neq 4$, $i_3 \neq 1$, and $i_4 \neq 1$. Let $x_1 = (e)^5$, $x_2 = ((x_1)^2)^5$, $x_3 = ((x_2)^3)^5$, and $x_4 = ((x_3)^4)^5$.

Assume that $i_3 = 3$. We set

$W_1 = \langle 12345, 32145, 42135, 12435, 21435, 14235, 24135 \rangle$, 
$W_2 = \langle 12345, 21345, 31245, 13245, 23145, 43125, 13425, 31425, 41325, 14325 \rangle$, and 
$W_3 = \langle 12345, 32415, 23415, 43215, 34215, 43125, 14325, 31425, 41325, 14325 \rangle$.

Let $u_1 = 24135$, $u_2 = 41325$, $u_3 = 34125$, and $u_4 = ((x_4)^3)^5$. Obviously, $x_1$ is a black vertex of $S_5^{(i)}$. By Lemma 4, there exists a hamiltonian path $Q_1$ of $S_5^{(i)} - \{x_2, (x_3)^3\}$ joining the white vertex $(u_1)^5$ to a black vertex $(y_1)_1 = i_1$. Again, there exists a hamiltonian path $Q_2$ of $S_5^{(i)} - \{x_4, (x_4)^3\}$ joining the black vertex $(u_2)^5$ to a white vertex $(y_2)_1 = i_2$. Moreover, there exists a hamiltonian path $Q_3$ of $S_5^{(i)} - \{x_3, (x_4)^4\}$ joining the black vertex $(u_3)^5$ to a white vertex $(y_3)_1 = i_3$. And, there exists a hamiltonian path.
Let \( Q_4 \) of \( S_5^{[2]} - \{x_2, (x_2)^3\} \) joining the black vertex \((u_4)^5\) to a white vertex \(y_4\) with \((y_4)_1 = i_4\). We set

\[
P_1 = \langle e, W_1, u_1, (u_1)^5, Q_1, y_1 \rangle,
\]
\[
P_2 = \langle e, W_2, u_2, (u_2)^5, Q_2, y_2 \rangle,
\]
\[
P_3 = \langle e, W_3, u_3, (u_3)^5, Q_4, y_4 \rangle. \quad \text{and}
\]
\[
P_4 = \langle e, x_1, (x_1)^2, x_2, (x_2)^3, x_3, (x_3)^4, x_4, (x_4)^3, u_4, Q_3, y_3 \rangle.
\]

Clearly, \( l(P_1) = 29 \) and \( l(P_2) = 30 \) for \( 2 \leq i \leq 4 \). Apparently, \( P_1, P_2, \ldots, P_{n-1} \) forms the desired paths.

Assume that \( i_1 \neq 3 \). Obviously, \( i_3 \neq 3 \). We set

\[
W_1 = \langle 12345, 21345, 14325, 13425, 34125, 43125, 13425, 31425 \rangle,
\]
\[
W_2 = \langle 12345, 32145, 23145, 13245, 41235, 42135, 21435, 42135 \rangle, \quad \text{and}
\]
\[
W_3 = \langle 12345, 42315, 24135, 34215, 32415, 23145, 32415, 12435, 21435 \rangle.
\]

Let \( u_1 = 31425, u_2 = 42135, u_3 = 21435, \) and \( u_4 = ((x_4)^3)^5 \). Obviously, \( x_1 \) is a black vertex of \( S_5^{[i]} \). By Lemma 4, there exists a hamiltonian path \( Q_1 \) of \( S_5^{[3]} - \{x_3, (x_3)^4\} \) joining the white vertex \((u_1)^5\) to a black vertex \( y_1 \) with \((y_1)_1 = i_1\). Again, there exists a hamiltonian path \( Q_2 \) of \( S_5^{[4]} - \{x_4, (x_4)^3\} \) joining the black vertex \((u_2)^5\) to a white vertex \( y_2 \) with \((y_2)_1 = i_2\). Moreover, there exists a hamiltonian path \( Q_3 \) of \( S_5^{[1]} - \{x_1, (x_1)^2\} \) joining the black vertex \((u_4)^5\) to a white vertex \( y_3 \) with \((y_3)_1 = i_3\). And, there exists a hamiltonian path \( Q_4 \) of \( S_5^{[2]} - \{x_2, (x_2)^3\} \) joining the black vertex \((u_3)^5\) to a white vertex \( y_4 \) with \((y_4)_1 = i_4\). We set

\[
P_1 = \langle e, W_1, u_1, (u_1)^5, Q_1, y_1 \rangle,
\]
\[
P_2 = \langle e, W_2, u_2, (u_2)^5, Q_2, y_2 \rangle,
\]
\[
P_3 = \langle e, x_1, (x_1)^2, x_2, (x_2)^3, x_3, (x_3)^4, x_4, (x_4)^3, u_4, Q_3, y_3 \rangle. \quad \text{and}
\]
\[
P_4 = \langle e, W_3, u_3, (u_3)^5, Q_4, y_4 \rangle.
\]

Clearly, \( l(P_1) = 29 \) and \( l(P_2) = 30 \) for \( 2 \leq i \leq 4 \). Apparently, \( P_1, P_2, \ldots, P_{n-1} \) forms the desired paths.

**Case 2:** \( n \geq 6 \). We set that \( x_j = (e)^j \) and \( y_j = ((e)^j)^{j-1} \) for \( 3 \leq j \leq n - 2 \). Obviously, \( u_j \) is a black vertex and \( v_j \) is a white vertex for \( 3 \leq j \leq n - 2 \).

Assume that \( i_1 \neq n - 3 \). By Lemma 5, there exists a hamiltonian path \( R \) of \( S_n^{[(n-1,n)]} - F_{n-2} \) joining \( e \) to a black vertex \( x_{j-1} \) with \((x_{j-1})_1 = 2\). We recursively set \( x_j \) as the unique neighborhood of \( (x_{j-1})^{n-1} \) in \( S_n^{[(j,n)]} \) with \((x_j)_1 = j + 1\) for \( 2 \leq j \leq n - 5 \), and we set \( x_{n-4} \) as the unique neighborhood of \( (x_{n-5})^{n-1} \) in \( S_n^{[(n-4,n)]} \) with \((x_{n-4})_1 = n - 1\). It is easy to see that \( x_j \) is a black vertex for \( 1 \leq j \leq n - 4 \) and \( \{(x_j)^{n-1}, x_{j+1}\} \in S_n^{[(j+1,n)]} \) for \( 1 \leq j \leq n - 5 \). Choose \( x_{n-3} \) as any black vertex in \( S_n^{n-1} \) with \((x_{n-3})_1 = n \) and
\((x_{n-3})_{n-1} = n - 3\). By Theorem 1, there exists a hamiltonian path \(P'\) of \(S_n^{(n-1)}\) joining the white vertex \((x_{n-4})^n\) to \(x_{n-3}\). Then we set the black vertex \(y_1\) as the unique neighborhood of the white vertex \((x_{n-3})^n\) in \(S_n^{(n-3,n)}\) with \((y_1)_{i_1} = i_1\). We set \(P_1\) as \((e, R, x_1, (x_1)^{n-1}, x_2, (x_2)^{n-1}, \ldots, x_{n-5}, (x_{n-5})^{n-1}, x_{n-4}, (x_{n-4})^n, R', x_{n-3}, (x_{n-3})^n, y_1)\). Obviously, \(l(P_1) = n(n - 2)! - 1\).

By Theorem 1, there exists a hamiltonian path \(H\) of \(S_n^{(1)}\) joining the black vertex \((e)^n\) to a white vertex \(z\) with \((z)_{i_1} = n\) and \((z)_{n-2} = n - 2\). Obviously, there exists a hamiltonian path \(H'\) of \(S_n^{(n-2,n)}\) joining the black vertex \((z)^n\) to a white vertex \(y_2\) with \((y_2)_{i_2} = a_2\). We set \(P_2\) as \((e, (e)^n, H, z, (z)^n, H', y_2)\). Obviously, \(l(P_2) = n(n - 2)!\).

By Theorem 1, there exists a hamiltonian path \(W\) of \(S_n^{(1,n)}\) joining the black vertex \((e)^{n-1}\) to a white vertex \(r\) with \((r)_{i_1} = n - 1\). Moreover, there exists a hamiltonian path \(W'\) of \(S_n^{(n-1)}\) joining the black vertex \((r)^n\) to a white vertex \(y_3\) with \((y_3)_{i_3} = a_3\). We set \(P_3\) as \((e, (e)^{n-1}, W, r, (r)^n, W', y_3)\). Obviously, \(l(P_3) = n(n - 2)!\).

Let \(i\) be any index with \(4 \leq i \leq n - 2\). By Lemma 4, there exists a hamiltonian path \(Q_{i}\) of \(S_n^{(i-2,n)}\) joining the black vertex \((v_{i-1})^{n-1}\) to a white vertex \(z_i\) with \((z_i)_{i_1} = i - 2\). Again, there exists a hamiltonian path \(Q'_{i}\) of \(S_n^{(i-2)}\) joining the black vertex \((z_i)^n\) to a white vertex \(y_i\) with \((y_i)_{i_1} = a_i\). Then we set \(P_i\) as \((e, u_{i-1}, v_{i-1}, (v_{i-1})^{n-1}, Q_i, z_i, (z_i)^n, Q'_i, y_i)\). Obviously, \(l(P_i) = n(n - 2)!\).

By Lemma 4, there exists a hamiltonian path \(R\) of \(S_n^{(n-3,3)}\) joining the black vertex \((v_{n-3})^{n-1}\) to a white vertex \(r\) with \((r)_{i_1} = n - 2\). Obviously, there exists a hamiltonian path \(R'\) of \(S_n^{(n-2)}\) joining the black vertex \((r)^n\) to a white vertex \(y_{n-1}\) with \((y_{n-1})_{i_1} = a_{n-1}\). Hence we set \(P_{n-1}\) as \((e, u_{n-1}, v_{n-1}, (v_{n-1})^{n-1}, R, r, (r)^n, R', y_{n-1})\). Obviously, \(l(P_{n-1}) = n(n - 2)!\).

 Apparently, \(P_1, P_2, \ldots, P_{n-1}\) form the desired paths. See Figure 2 for illustration for the case \(n = 6\).

Assume that \(i_1 = n - 3\). By Lemma 5, there exists a hamiltonian path \(R\) of \(S_n^{(n-1,1)}\) joining \(e\) to a black vertex \(x_1\) with \((x_1)_{i_1} = n - 3\). We recursively set \(x_j\) as the unique neighborhood of \((x_{j-1})^{n-1}\) in \(S_n^{(j-1,n)}\) with \((x_j)_{i_1} = n - j - 1\) for \(2 \leq j \leq n - 5\), and we set \(x_{n-4}\) as the unique neighborhood of \((x_{n-5})^{n-1}\) in \(S_n^{(3,n)}\) with \((x_{n-4})_{i_1} = n - 1\). It is easy to see that \(x_j\) is a black vertex for \(1 \leq j \leq n - 4\) and \((x_j)^{n-1}, x_{j+1}\) in \(S_n^{(n-j-2,n)}\) for \(1 \leq j \leq n - 5\). By Theorem 1, there exists a hamiltonian path \(R'\) of \(S_n^{(n-1)}\) joining the white vertex \((x_{n-4})^n\) to a black vertex \(x_{n-3}\) with \((x_{n-3})_{i_1} = n\) and \((x_{n-3})_{n-1} = 2\). Then we set the black vertex \(y_1\) as the unique neighborhood of the white vertex \((x_{n-3})^n\) in \(S_n^{(2,2)}\) with \((y_1)_{i_1} = i_1\). We set \(P_1\) as \((e, R, x_1, (x_1)^{n-1}, x_2, (x_2)^{n-1}, \ldots, x_{n-4}, (x_{n-4})^n, R', x_{n-3}, (x_{n-3})^n, y_1)\). Obviously, \(l(P_1) = n(n - 2)! - 1\).

By Theorem 1, there exists a hamiltonian path \(H\) of \(S_n^{(1)}\) joining the black vertex \((e)^n\) to a white vertex \(z\) with \((z)_{i_1} = n\) and \((z)_{n-2} = n - 2\). Again, there exists a hamiltonian
path $H'$ of $S_n^{(n-2,n)}$ joining the black vertex $(z)^n$ to a white vertex $y_2$ with $(y_2)_1 = a_2$. We set $P_2$ as \( \langle e, (e)^n, H, z, (z)^n, H', y_2 \rangle \). Obviously, $l(P_2) = n(n-2)!$.

By Theorem 1, there exists a hamiltonian path $W$ of $S_n^{(1,n)}$ joining the black vertex $(e)^{n-1}$ to a white vertex $r$ with $(r)_1 = n-1$. Moreover, there exists a hamiltonian path $W'$ of $S_n^{(n-1)}$ joining the black vertex $(r)^n$ to a white vertex $y_3$ with $(y_3)_1 = i_3$. We set $P_3$ as \( \langle e, (e)^{n-1}, W, r, (r)^n, W', y_3 \rangle \). Obviously, $l(P_3) = n(n-2)!$.

Let $j$ be any index with $4 \leq j \leq n-2$. By Lemma 4, there exists a hamiltonian path $Q_j$ of $S_n^{(j-1,n)} = \{(x_{n-j-1})^{n-1}, x_{n-j}\}$ joining the black vertex $(z_j)^n$ to a white vertex $y_j$ with $(y_j)_1 = i_j$. Then we set $P_j$ as \( \langle e, u_j, v_j, (v_j)^{n-1}, Q_j, z_j, (z_j)^n, Q_j', y_j \rangle \). Obviously, $l(P_j) = n(n-2)!$.

By Lemma 4, there exists a hamiltonian path $R$ of $S_n^{(2,n)} = \{(x_{n-3})^n, y_1\}$ joining the black vertex $(v_3)^{n-1}$ to a white vertex $r$ with $(r)_1 = n-2$. Obviously, there exists a hamiltonian path $R'$ of $S_n^{(n-2)}$ joining the black vertex $(r)^n$ to a white vertex $y_{n-1}$ with $(y_{n-1})_1 = i_{n-1}$. Hence we set $P_{n-1}$ as \( \langle e, u_3, v_3, (v_3)^{n-1}, R, r, (r)^n, R', y_{n-1} \rangle \). Obviously, $l(P_{n-1}) = n(n-2)!$.
Apparently, \( P_1, P_2, \ldots, P_{n-1} \) forms the desired paths. See Figure 3 for illustration for the case \( n = 6 \). The lemma is proved. □

![Figure 3: Illustration for the case 2 of Lemma 9.](image)

**Lemma 10** \( D_{n-1}^{SL}(S_n) \geq \frac{n!}{n-2} + 1 = (n-1)! + 2(n-2)! + 2(n-2)! + 1 \).

**Proof.** Let \( u \) and \( v \) are two adjacent vertices of \( S_n \). Let \( P_1, P_2, \ldots, P_{n-1} \) be any \((n-1)^*\)-container of \( S_n \) joining \( u \) to \( v \). Obviously, one of these path is \( \langle u, v \rangle \). Thus, \( \max \{ l(P_i) \mid 1 \leq i \leq n-1 \} \geq \frac{n!}{n-2} + 1 \). Hence, \( d_{n-1}^{SL}(u, v) \geq \frac{n!}{n-2} + 1 \) and \( D_{n-1}^{SL}(S_n) \geq \frac{n!}{n-2} + 1 \). □

**Lemma 11** \( D_4^{SL}(S_5) \leq 41 \).

**Proof.** Let \( u \) be any white vertex and \( v \) be any black vertex of \( S_5 \). Obviously, \( d(u, v) \) is odd.

**Case 1:** \( d(u, v) = 1 \). Since the \( S_5 \) is vertex transitive and edge transitive, we may assume that \( u = e = 12345 \) and \( v = (e)^5 = 52341 \). By Lemma 7, there exist \( P_1, P_2, \) and \( P_3 \) of \( S_5^{[5]} \) such that (1) \( P_1 \) joins 12345 to the black vertex 24135 with \( l(P_1) = 7 \), (2) \( P_2 \) joins 12345 to the white vertex 34125 with \( l(P_2) = 8 \), (3) \( P_3 \) joins 12345 to the white vertex 41325 with \( l(P_3) = 8 \), and (4) \( P_1 \cup P_2 \cup P_3 \) spans \( S_5^{[5]} \). Similarly, there exist \( Q_1, Q_2, \) and \( Q_3 \) of \( S_5^{[1]} \).
such that (1) $Q_1$ joins 52341 to the white vertex 24531 with $l(Q_1) = 7$, (2) $Q_2$ joins 52341 to the black vertex 34521 with $l(Q_2) = 8$, (3) $Q_3$ joins 52341 to the black vertex 45321 with $l(Q_3) = 8$, and (4) $Q_1 \cup Q_2 \cup Q_3$ spans $S_4^{(1)}$. By Theorem 1, there is a hamiltonian path $R_1$ of $S_5^{(2)}$ joining the white vertex 54132 to the black vertex 14532, there is a hamiltonian path $R_2$ of $S_5^{(3)}$ joining the black vertex 34125 to the white vertex 41324, and there is a hamiltonian path $R_3$ of $S_5^{(4)}$ joining the black vertex 51324 to the white vertex 15424. Then we set

$$T_1 = (12345, P_1, 24135, 54132, R_1, 14532, 24531, (Q_1)^{-1}, 52341),$$
$$T_2 = (12345, P_2, 34125, 54123, R_2, 14523, 34521, (Q_2)^{-1}, 52341),$$
$$T_3 = (12345, P_3, 41325, 51324, R_3, 15324, 45321, (Q_3)^{-1}, 52341),$$
$$T_4 = (12345, 52341).$$

Obviously, $\{T_1, T_2, T_3, T_4\}$ forms a 4*-container of $S_5$ between $e$ and $(e)^5$. Moreover, $l(T_1) = 39$, $l(T_2) = l(T_3) = 41$, and $l(T_4) = 1$. Thus, $d_4^*(e, (e)^5) \leq 41$.

**Case 2:** $d(u, v) \geq 3$. Since $d(u, v) \geq 3$, we may assume that $u = e$, $(v)_5 = 4$, and $(v)_1 \in \{1, 2, 3\}$.

**Subcase 2.1:** $(v)_1 = 1$. By Lemma 8, there exist four paths $P_1, P_2, P_3$, and $P_4$ of $S_5^{(5)}$ such that (1) $P_1$ joins $e$ to a white vertex $w$ with $(w)_1 = 2$ and $l(P_1) = 2$, (2) $P_2$ joins $e$ to a white vertex $x$ with $(x)_1 = 3$ and $l(P_1) = 2$, (3) $P_3$ joins $e$ to a black vertex $y$ with $(y)_1 = 4$ and $l(P_3) = 19$, (4) $P_4$ joins $e$ to a black vertex $z \neq y$ with $(z)_1 = 4$ and $l(P_4) = 19$, (5) $P_1 \cup P_2 \cup P_3$ spans $S_5^{(5)}$, and (6) $P_1 \cup P_2 \cup P_3$ spans $S_5^{(5)}$.

Similarly, there exist four paths $Q_1, Q_2, Q_3$, and $Q_4$ of $S_5^{(4)}$ such that (1) $Q_1$ joins $v$ to a black vertex $p$ with $(p)_1 = 2$ and $l(Q_1) = 2$, (2) $Q_2$ joins $v$ to a black vertex $q$ with $(q)_1 = 3$ and $l(Q_2) = 2$, (3) $Q_3$ joins $v$ to a white vertex $r$ with $(r)_1 = 5$ and $l(Q_3) = 19$, (4) $Q_4$ joins $v$ to a white vertex $s \neq r$ with $(s)_1 = 5$ and $l(Q_4) = 19$, (5) $Q_1 \cup Q_2 \cup Q_3$ spans $S_5^{(4)}$, and (6) $Q_1 \cup Q_2 \cup Q_4$ spans $S_5^{(4)}$.

By Lemma 1, there are exactly three edges joining a black vertex of $S_5^{(5)}$ to a white vertex of $S_5^{(4)}$. By pigeon-hole principle, at least one vertex in $\{y, z\}$ is adjacent to a vertex in $\{r, s\}$. Without loss of generality, we assume that $y$ is adjacent to $r$. Let $T_1$ be the hamiltonian path of $S_5^{(1)}$ joining the black vertex $(e)^5$ to the white vertex $(v)^5$, $T_2$ be the hamiltonian path of $S_5^{(2)}$ joining the black vertex $(w)^5$ to the white vertex $(p)^5$, and $T_3$ be the hamiltonian path of $S_5^{(3)}$ joining the black vertex $(x)^5$ to the white vertex $(q)^5$. We set

$$H_1 = \langle e, (e)^5, T_1, (v)^5, v \rangle,$$
$$H_2 = \langle e, P_1, w, (w)^5, T_2, (p)^5, p, Q_1^{-1}, v \rangle,$$
$$H_3 = \langle e, P_2, x, (x)^5, T_3, (q)^5, q, Q_2^{-1}, v \rangle,$$
$$H_4 = \langle e, P_3, y, r, Q_3^{-1}, v \rangle.$$
Obviously, \( \{H_1, H_2, H_3, H_4\} \) forms a \( 4^*-\)container of \( S_5 \) between \( e \) and \( v \). Moreover, 
\[ l(H_1) = 25, \ l(H_2) = l(H_3) = 29, \text{ and } l(H_4) = 39. \] Thus, \( d^L_4(e, v) \leq 41. \)

**Subcase 2.2: \( (v)_1 = a \in \{2, 3\} \).** Let \( b \) be the only element in \( \{2, 3\} - \{a\} \). By Lemma 8, there exist four paths \( P_1, P_2, P_3, \) and \( P_4 \) of \( S_5^{(3)} \) such that (1) \( P_1 \) joins \( e \) to a white vertex \( w \) with \( (w)_1 = a \) and \( l(P_1) = 2 \), (2) \( P_2 \) joins \( e \) to a white vertex \( x \) with \( (x)_1 = b \) and \( l(P_2) = 2 \), (3) \( P_3 \) joins \( e \) to a black vertex \( y \) with \( (y)_1 = 4 \) and \( l(P_3) = 19 \), (4) \( P_4 \) joins \( e \) to a black vertex \( z \neq y \) with \( (z)_1 = 4 \) and \( l(P_4) = 19 \). (5) \( P_1 \cup P_2 \cup P_3 \) spans \( S_5^{(3)} \), and (6) \( P_1 \cup P_2 \cup P_4 \) spans \( S_5^{(3)} \).

Again, there exist four paths \( Q_1, Q_2, Q_3, \) and \( Q_4 \) of \( S_5^{(4)} \) such that (1) \( Q_1 \) joins \( v \) to a black vertex \( p \) with \( (p)_1 = 1 \) and \( l(Q_1) = 2 \), (2) \( Q_2 \) joins \( v \) to a black vertex \( q \) with \( (q)_1 = b \) and \( l(Q_2) = 2 \), (3) \( Q_3 \) joins \( v \) to a white vertex \( r \) with \( (r)_1 = 5 \) and \( l(Q_3) = 19 \), (4) \( Q_4 \) joins \( v \) to a white vertex \( s \neq r \) with \( (s)_1 = 5 \) and \( l(Q_4) = 19 \). (5) \( Q_1 \cup Q_2 \cup Q_3 \) spans \( S_5^{(4)} \), and (6) \( Q_1 \cup Q_2 \cup Q_4 \) spans \( S_5^{(4)} \).

By Lemma 1, there are exactly three edges joining a black vertex of \( S_5^{(5)} \) to a white vertex of \( S_5^{(4)} \). By pigeon-hole principle, at least one vertex in \( \{y, z\} \) is adjacent to a vertex in \( \{r, s\} \). Without loss of generality, we assume that \( y \) is adjacent to \( r \). Let \( T_1 \) be the hamiltonian path of \( S_5^{(1)} \) joining the black vertex \( e \) to the white vertex \( p \), \( T_2 \) be the hamiltonian path of \( S_5^{(a)} \) joining the black vertex \( w \) to the white vertex \( q \), and \( T_3 \) be the hamiltonian path of \( S_5^{(b)} \) joining the black vertex \( x \) to the white vertex \( r \). We set
\[
\begin{align*}
H_1 &= \langle e, (e)^5, T_1, (p)^5, p, Q_1^{-1}, v \rangle, \\
H_2 &= \langle e, P_1, w, (w)^5, T_2, (v)^5, v \rangle, \\
H_3 &= \langle e, P_2, x, (x)^5, T_3, (q)^5, q, Q_2^{-1}, v \rangle, \text{ and} \\
H_4 &= \langle e, P_3, y, r, Q_3^{-1}, v \rangle.
\end{align*}
\]

Obviously, \( \{H_1, H_2, H_3, H_4\} \) forms a \( 4^*-\)container of \( S_5 \) between \( e \) and \( v \). Moreover, 
\[ l(H_1) = l(H_2) = 27, l(H_3) = 29, \text{ and } l(H_4) = 39. \] Thus, \( d^L_4(e, v) \leq 41. \)

**Lemma 12** \( d^L_{n-1}(u, v) \leq (n - 1)! + 2(n - 2)! + 2(n - 3)! + 1 = \frac{n!}{n-2} + 1 \) for \( n \geq 6 \).

**Proof.** Let \( u \) be any white vertex and \( v \) be any black vertex of \( S_n \). Obviously, \( d(u, v) \) is odd.

**Case 1:** \( d(u, v) = 1 \). Since the star graph is vertex transitive and edge transitive, we may assume that \( u = e \) and \( v = (e)^n \).

By Lemma 9, there exist \( (n - 2) \) paths \( P_1, P_2, \ldots, P_{n-2} \) of \( S_n^{(n)} \) such that (1) \( P_1 \) joins \( e \) to a black vertex \( x_1 \) with \( (x_1)_1 = 2 \) and \( P_1 = n(n - 2)! - 1 \), (2) \( P_i \) joins \( e \) to a white vertices \( x_i \) with \( (x_i)_1 = i + 1 \) and \( l(P_i) = n(n - 2)! \) for \( 2 \leq i \leq n - 2 \), and (3) \( \cup_{i=1}^{n-2} P_i \) spans \( S_n^{(n)} \). And there exist \( (n - 2) \) paths \( Q_1, Q_2, \ldots, Q_{n-2} \) of \( S_n^{(1)} \) such that (1) \( Q_1 \) joins \( (e)^n \) to a white vertices \( y_1 \) with \( (y_1)_1 = 2 \) and \( l(Q_1) = n(n - 2)! - 1 \), (2) \( Q_i \) joins \( (e)^n \) to a black vertices \( y_i \)
with \((y_1)_i = i + 1\) and \(l(Q_i) = n(n-2)!\) for \(2 \leq i \leq n - 2\), and \((3) \cup_{i=2}^{n-2} Q_i \) spans \(S_{n}^{(1)}\). By Theorem 1, there exists a hamiltonian path \(R_1\) of \(S_{n}^{(2)}\) joining the white vertex \((x_1)_n\) to the black vertex \((y_1)_n\) and there exists a hamiltonian path \(R_i\) of \(S_{n}^{(i+1)}\) joining the black vertex \((x_i)_n\) to the white vertex \((y_i)_n\) for \(2 \leq i \leq n - 3\).

We set \(P_i = (e, P_i, x_i, (x_i)_n, R_i, (y_i)_n, y_i, Q_i^{-1}, (e)_n)\) for \(1 \leq i \leq n - 2\) and \(P_{n-1} = (e, (e)_n)\). Then \(P_1, P_2, \ldots, P_{n-1}\) forms an \((n-1)^*\)-container between \(e\) and \((e)_n\). Obviously, \(l(P_i) = (n-1)! + 2(n-2)! + 2(n-3)! - 1\), \(l(P_1) = (n-1)! + 2(n-2)! + 2(n-3)! + 1\) for \(2 \leq i \leq n - 2\), and \(l(P_{n-1}) = 1\). Hence \(d_{n-1}^{E}(e, (e)_n) \leq (n-1)! + 2(n-2)! + 2(n-3)! + 1\).

**Case 2:** \(d(u, v) \geq 3\). Since \(d(u, v) \geq 3\), we may assume that \(u = e\), \((v)_n = n - 1\), and \((v)_1 \neq n\).

**Subcase 2.1:** \((v)_1 = 1\). By Lemma 9, there exist \((n-2)\) paths \(P_1, P_2, \ldots, P_{n-2}\) of \(S_{n}^{(n)}\) such that \(1) P_1 \) joins \(e\) to a black vertex \(x_1\) with \((x_1)_1 = 1\) and \(l(P_1) = n(n-2)! - 1\), \(2) P_i \) joins \(e\) to a white vertex \(x_i\) with \((x_i)_1 = i\) and \(l(P_i) = n(n-2)!\) for \(2 \leq i \leq n - 2\), and \(3) \cup_{i=2}^{n-2} P_i \) spans \(S_{n}^{(n)}\). Again there exist \((n-2)\) paths \(Q_1, Q_2, \ldots, Q_{n-2}\) of \(S_{n}^{(n-1)}\) such that \(1) Q_1 \) joins \(v\) to a white vertex \(y_1\) with \((y_1)_1 = 1\) and \(l(Q_1) = n(n-2)! - 1\), \(2) Q_i \) joins \(v\) to a black vertex \(y_i\) with \((y_i)_1 = i\) and \(l(Q_i) = n(n-2)!\) for \(2 \leq i \leq n - 2\), and \(3) \cup_{i=1}^{n-2} Q_i \) spans \(S_{n}^{(n-1)}\).

By Lemma 6, there exist two disjoint paths \(H_1\) and \(H_2\) of \(S_{n}^{(1)}\) such that \(1) H_1 \) joins the white vertex \((x_1)_n\) to the black vertex \((y_1)_n\), \(2) H_2 \) joins the white vertex \((e)_n\) to the black vertex \((y_1)_n\), \(3) H_1 \cup H_2 \) spans \(S_{n}^{(1)}\). For \(2 \leq i \leq n - 2\), there exists a hamiltonian path \(R_i\) of \(S_{n}^{(i)}\) joining the black vertex \((x_i)_n\) to the white vertex \((y_i)_n\). We set

\[
T_1 = \langle e, P_1, x_1, (x_1)_n, H_1, (y_1)_n, y_1, Q_1^{-1}, v \rangle,
\]

\[
T_i = \langle e, P_i, x_i, (x_i)_n, R_i, (y_i)_n, y_i, Q_i^{-1}, v \rangle \text{ for } 2 \leq i \leq n - 2, \text{ and}
\]

\[
T_{n-1} = \langle e, (e)_n, H_1, (v)_n, v \rangle.
\]

Obviously, \(\{T_1, T_2, \ldots, T_{n-1}\}\) forms an \((n-1)^*\)-container. Moreover, \(l(T_i) \leq (n-1)! + 2(n-2)! + 2(n-3)! + 1\). Thus, \(d_{n-1}^{E}(e, (v)_n) \leq (n-1)! + 2(n-2)! + 2(n-3)! + 1\).

**Subcase 2.2:** \((v)_1 = t \neq 1, n\). By Lemma 9, there exist \((n-2)\) paths \(P_1, P_2, \ldots, P_{n-2}\) of \(S_{n}^{(n)}\) such that \(1) P_1 \) joins \(e\) to a black vertex \(x_1\) with \((x_1)_1 = 1\) and \(l(P_1) = n(n-2)! - 1\), \(2) P_i \) joins \(e\) to a white vertex \(x_i\) with \((x_i)_1 = i\) and \(l(P_i) = n(n-2)!\) for \(2 \leq i \leq n - 2\), and \(3) \cup_{i=2}^{n-2} P_i \) spans \(S_{n}^{(n)}\). Again there exist \((n-2)\) paths \(Q_1, Q_2, \ldots, Q_{n-2}\) of \(S_{n}^{(n-1)}\) such that \(1) Q_1 \) joins \(v\) to a white vertex \(y_1\) with \((y_1)_1 = 1\) and \(l(Q_1) = n(n-2)! - 1\), \(2) Q_i \) joins \(v\) to a black vertex \(y_i\) with \((y_i)_1 = i\) and \(l(Q_i) = n(n-2)!\) for \(2 \leq i \leq n - 2\), and \(3) \cup_{i=1}^{n-2} Q_i \) spans \(S_{n}^{(n-1)}\).

Let the black vertex \(w\) be the unique neighbor of the white vertex \((v)_n\) of \(S_{n}^{(t)}\) with \((w)_1 = 1\). By Lemma 6, there exist two disjoint paths \(H_1\) and \(H_2\) of \(S_{n}^{(1)}\) such that \(1) H_1 \) joins the white vertex \((x_1)_n\) to the black vertex \((y_1)_n\), \(2) H_2 \) joins the white vertex \((e)_n\) to the black vertex \((y_1)_n\), \(3) H_1 \cup H_2 \) spans \(S_{n}^{(1)}\). We set

\[
T_1 = \langle e, P_1, x_1, (x_1)_n, H_1, (y_1)_n, y_1, Q_1^{-1}, v \rangle,
\]

\[
T_i = \langle e, P_i, x_i, (x_i)_n, R_i, (y_i)_n, y_i, Q_i^{-1}, v \rangle \text{ for } 2 \leq i \leq n - 2, \text{ and}
\]

\[
T_{n-1} = \langle e, (e)_n, H_1, (v)_n, v \rangle.
\]

Obviously, \(\{T_1, T_2, \ldots, T_{n-1}\}\) forms an \((n-1)^*\)-container. Moreover, \(l(T_i) \leq (n-1)! + 2(n-2)! + 2(n-3)! + 1\). Thus, \(d_{n-1}^{E}(e, (v)_n) \leq (n-1)! + 2(n-2)! + 2(n-3)! + 1\).
the black vertex \((w)^n\), and (3) \(H_1 \cup H_2\) span \(S_n^{(1)}\).

By Theorem 4, there exists a hamiltonian path \(R_t\) of \(S_n^{(t)}\) joining the black vertex \((x_t)^n\) to the white vertex \((y_t)^n\). By Theorem 1, there exists a hamiltonian path \(R_i\) of \(S_n^{(t)}\) joining the black vertex \((x_i)^n\) to the white vertex \((y_i)^n\) for \(2 \leq i \leq n - 2\) with \(i \neq t\). We set

\[
T_1 = \langle e, P_1, x_1, (x_1)^n, H_1, (y_1)^n, y_1, Q_i^{-1}, v \rangle, \\
T_i = \langle e, P_i, x_i, (x_i)^n, R_i, (y_i)^n, y_i, Q_i^{-1}, v \rangle \text{ for } 2 \leq i \leq n - 2, \text{ and} \\
T_{n-1} = \langle e, (e)^n, H_2, (w)^n, w, (v)^n, v \rangle.
\]

Obviously, \(\{T_1, T_2, \ldots, T_{n-1}\}\) forms an \((n - 1)^\ast\)-container. Moreover, \(l(T_i) \leq (n - 1)! + 2(n - 2)! + 2(n - 3)! + 1\). Thus, \(d_{n-1}^{2L}(e, v) \leq (n - 1)! + 2(n - 2)! + 2(n - 3)! + 1\).

\[\square\]

It is easy to check that \(D_{1}^{2L}(S_2) = 1\) and \(D_{2}^{2L}(S_3) = 3\). Using computer, we have that \(D_{3}^{2L}(S_4) = 15\) by brute force. By Lemmas 10, 11, and 12, we have the following theorem:

**Theorem 5** \(D_{n-1}^{2L}(S_n) = (n - 1)! + 2(n - 2)! + 2(n - 3)! \text{ for } n \geq 5\).

### 4 The 2\(2^L\)-diameter \(S_n\)

**Lemma 13** Assume that \(u\) be any white vertex of \(S_4\). Let \(a\) and \(b\) are two distinct elements of \(\langle 4 \rangle\). There exist two paths \(P_1\) and \(P_2\) of \(S_4\) such that (1) \(P_1\) joins \(u\) to a black vertex \(x\) with \((x)_1 = a\) and \(l(P_1) = 7\), (2) \(P_2\) joins \(u\) to a white vertex \(y\) with \((y)_1 = b\) and \(l(P_2) = 16\), and (3) \(P_1 \cup P_2\) spans \(S_4\).

**Proof.** Without loss of generality, we may assume that \(u = e\). The required two paths are listed below.

\[
\begin{align*}
\end{align*}
\]

\[\square\]

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Theorem 6

\[ D^{s_L}_{2}(S_n) = \begin{cases} 
3 & \text{if } n = 3, \\
15 & \text{if } n = 4, \text{ and} \\
\frac{n^2}{2} + 1 & \text{if } n \geq 5. 
\end{cases} \]

Proof. It is easy to check that \( D^{s_L}_{2}(S_3) = 3 \). Using computer, we have that \( D^{s_L}_{2}(S_4) = 15 \) by brute force. Thus, we assume that \( n \geq 5 \). Let \( u \) be a white vertex and \( v \) be a black vertex of \( S_n \). Let \( P_1 \) and \( P_2 \) be any two \( 2^s \)-container of \( S_n \) joining \( u \) to \( v \). Obviously, \( \max\{l(P_1), l(P_2)\} \geq \frac{n^2}{2} + 1 \). Hence, \( d^{s_L}_{2}(u, v) \geq \frac{n^2}{2} + 1 \) and \( D^{s_L}_{2}(S_n) \geq \frac{n^2}{2} + 1 \). Hence we only need to show that \( d^{s_L}_{2}(u, v) \leq \frac{n^2}{2} + 1 \). Without loss of generality, we assume that \( u = e \) and \( v \in S_n^{(n-1)} \).

Case 1: \( n = 5 \). By Lemma 13, there exist two paths \( H_1 \) and \( H_2 \) of \( S_5^{[5]} \) such that (1) \( H_1 \) joins \( e \) to a black vertex \( x \) with \( (x)_1 = 1 \) and \( l(H_1) = 7 \), (2) \( H_2 \) joins \( e \) to a white vertex \( y \) with \( (y)_1 = 3 \) and \( l(H_2) = 16 \), and (3) \( H_1 \cup H_2 \) spans \( S_5^{[5]} \). Again, there exist two paths \( T_1 \) and \( T_2 \) of \( S_5^{[4]} \) such that (1) \( T_1 \) joins \( v \) to a white vertex \( p \) with \( (p)_1 = 2 \) and \( l(T_1) = 7 \), (2) \( T_2 \) joins \( v \) to a black vertex \( q \) with \( (q)_1 = 3 \) and \( l(T_2) = 16 \), and (3) \( T_1 \cup T_2 \) spans \( S_5^{[4]} \). By Lemma 3, there exists a hamiltonian path \( W \) of \( S_5^{[1, 2]} \) joining the white vertex \( (x)_5 \) to the black vertex \( (p)_5 \). Moreover, there exists a hamiltonian path \( Z \) of \( S_5^{[3]} \) joining the black vertex \( (y)_5 \) to the white vertex \( (q)_5 \). We set

\[
L_1 = \langle e, H_1, x, (x)_n, W, (p)_n, p, T_1^{-1}, v \rangle \quad \text{and} \quad L_2 = \langle e, H_2, y, (y)_n, Z, (q)_n, q, T_2^{-1}, v \rangle.
\]

Obviously, \( L_1 \) and \( L_2 \) forms a \( 2^s \)-container. Moreover, \( l(L_1) = 61 \) and \( l(L_2) = 59 \). Hence \( d^{s_L}_{2}(u, v) \leq \frac{n^2}{2} + 1 \).

Case 2: \( n \geq 6 \) is even. Let \( x \) be any neighborhood of \( v \) in \( S_n^{(n-1)} \) with \( (x)_1 \neq n \). Obviously, \( x \) is a white vertex. Let \( j \) be any index in \( \{n\} - \{ (x)_1, n - 1 \} \). Obviously, \( (e)^j \) is a black vertex. Let \( a_1, a_2, \ldots, a_{n-2} \) be a permutation of \( \{n-2\} \) such that \( a_k = j \) and \( (x)_1 = a_3 \). By Theorem 2, there exists a hamiltonian path \( P \) of \( S_n^{[n]} - \{ (e)^j \} \) joining \( e \) to a white vertex \( p \) with \( (p)_1 = a_1 \) and \( (p)_2 = a_{n-2} \). By Lemma 4, there exists a hamiltonian path \( H \) of \( S_n^{[a_1]} - \{ (p)^n, (p)^n, 2 \} \) joining the white vertex \( (e)^j \) to a black vertex \( y \) with \( (y)_1 = a_3 \). Let \( H = \{ a_1, a_2, \ldots, a_{n-2} \} \) and \( H' = \{ n - 2 \} - H \). By Lemma 3, there exists a hamiltonian path \( T \) of \( S_n^{[H-a_1]} \) joining the white vertex \( (y)^n \) to the black vertex \( (x)^n \). By Theorem 1, there exists a hamiltonian path \( H' \) of \( S_n^{[a_{n-2}]} \) joining the black vertex \( ((p)^n)^2 \) to a white vertex \( q \) with \( (q)_1 = n - 3 \) and \( (q)^n \neq v \). By Theorem 2, there exists a hamiltonian path \( T' \) of \( S_n^{[n-1]} - \{ x \} \) joining the black vertex \( (q)^n \) to \( v \). We set

\[
L_1 = \langle e, (e)^j, ((e)^j)^n, H, y, (y)^n, T, (x)^n, x, v \rangle \quad \text{and} \quad L_2 = \langle e, P, p, (p)^n, ((p)^n)^2, ((p)^n)^2, H', q, (q)^n, T', v \rangle.
\]

Obviously, \( L_1 \) and \( L_2 \) forms a \( 2^s \)-container. Moreover, \( l(L_1) = \frac{n^2}{2} + 1 \) and \( l(L_2) = \frac{n^2}{2} - 1 \). Hence \( d^{s_L}_{2}(u, v) \leq \frac{n^2}{2} + 1 \). See Figure 4 for illustration the case \( n = 6 \).
Case 3: $n \geq 7$ is odd. Let $x$ be any neighborhood of $v$ in $S^{(n-1)}_n$ with $(x)_1 \neq n$. Obviously, $x$ is a white vertex. Let $j$ be any index in $\{n\} - \{(x)_1, n - 1\}$. Obviously, $(e)_j$ is a black vertex. Let $a_1, a_2, \ldots, a_{n-2}$ be a permutation of $(n - 2)$ such that $a_1 = j$ and $(x)_1 = a_2$. By Theorem 2, there exists a hamiltonian path $P$ of $S^{(n)}_n - \{(e)_j\}$ joining $e$ to a white vertex $p$ with $(p)_1 = a_1$ and $(p)_2 = a_{n-3}$. By Lemma 4, there exists a hamiltonian path $R$ of $S^{(a_1)}_n - \{(p)^n, ((p)^n)^2\}$ joining the white vertex $(e)_j$ to a black vertex $y$ with $(y)_1 = a_{n-2}$ and $(y)_{n-1} = a_2$. Let $X$ be the subgraph of $S_n$ induced by $\bigcup_{i=2}^{n-1} S^{(a_i,a_n)}_n$ and $X'$ be the subgraph of $S_n$ induced by $\bigcup_{i=2}^{n-3} S^{(a_i,a_n)}_n \cup S^{(n-1,a_n)}_n \cup S^{(n,a_n)}_n$. By Lemma 3, there exists a hamiltonian path $W$ of $X$ joining the white vertex $(y)_n$ to a black vertex $z$ with $(z)_1 = a_3$. Let $H = \{a_1, a_2, \ldots, a_{n-1}\}$ and $H' = \langle n - 3 \rangle - H$. Again, there exists a hamiltonian path $T$ of $S^{H-(a_1)}_n$ joining the white vertex $(z)_n$ to the black vertex $(x)_n$. By Theorem 3, there exists a hamiltonian path $R'$ of $S^{(H')}_n$ joining the black vertex $((p)^n)^2$ to a white vertex $q$ with $(q)_1 = a_{n-2}$ and $(q)_{n-1} = n$. By Lemma 3, there exists a hamiltonian path $W'$ of $Y$ joining the white vertex $(q)_n$ to a black vertex $r$ with $(r)_1 = n - 1$ and $r \neq v$. By Theorem 2, there exists a hamiltonian path $T'$ of $S^{(n-1)}_n - \{x\}$ joining the black vertex $(r)_n$ to the black vertex $v$. We set

$$L_1 = \langle e, (e)_j, ((e)_j)^n, R, y, (y)_n, W, z, (z)_n, T, (x)_n, x, v \rangle$$
and

$$L_2 = \langle e, p, (p)^n, ((p)^n)^2, ((y)_n)^2, R', q, (q)_n, W', r, (r)_n, T', v \rangle.$$

Obviously, $L_1$ and $L_2$ forms a $2^*$-container. Moreover, $l(L_1) = \frac{n^2}{2} + 1$ and $l(L_2) = \frac{n^2}{2} - 1$. Hence $d_2^*(u,v) \leq \frac{n^2}{2} + 1$. See Figure 5 for illustration the case $n = 7$. □
Figure 5: Illustration for the case 3 of Theorem 6.

5 Conclusion

In this paper, we prove that $D_{n-1}^S(S_n) = (n-1)! + 2(n-2)! + 2(n-3)! + 1 = \frac{n!}{n-2} + 1$ and $D_2^S(S_n) = \frac{n!}{2} + 1$ for $n \geq 5$. Actually, we prove that $d_2^S(u, v) = \frac{n!}{2} + 1$ for any two vertices $u$ and $v$ from different bipartite set of $S_n$.

Recently, we have proved that $S_n$ is super spanning laceable [10]. Hence we will study $D_k^S(S_n)$ for $3 \leq k \leq n-1$. We conjecture that $D_k^S(S_n) = \frac{n!}{k} + 1$ for $n \geq 5$ and $2 \leq k \leq n-2$.

References


