Lecture 3:

Review of Basic Circuit Analysis
(in frequency domain)

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Overview

• **Reading**
  – S&S:

• **Supplemental Reading**
  – Nilsson: Chapters 12-13

• **Background**
  – So far, we have look at how circuits respond to constant signals. In this section, we will consider how circuits respond to time varying signals. We will review the basic properties of a powerful tool called Laplace transforms that enables us to convert between time and frequency domain representation of circuits. The Laplace transform will be an analytical tool that we will use extensively to analyze complex circuits throughout the rest of the course.
Temporal Representation of Signals
Sinusoidal Source

- There are a set of parameters that determine the characteristics of a sinusoidal source. Let’s look at a sinusoidally varying voltage governed by the following equation...

  \[ v = V_m \cos(\omega t + \phi) \quad \omega = 2\pi f = \frac{2\pi}{T} \quad \text{(radians per second)} \]

- The phase angle \(\phi\) determines the value of the sinusoid at \(t = 0\)
  - The amount of shift (in time) is \(\phi/\omega\)
- Another important characteristic of the sinusoid is its rms (root mean square) value defined as

  \[ V_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) \, dt} = \frac{V_m}{\sqrt{2}} \]
Sinusoidal Response

- Look at the sinusoidal response of an RL circuit to a sinusoidal source

\[ v_s = V_m \cos(\omega t + \phi) \]

- From KVL,

\[ L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi) \]

- and the solution to this differential equation is

\[ i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta)e^{-(R/L)t} + \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta) \]

- where \( \theta \) is the angle whose tangent is \( \omega L/R \) or \( \tan \theta = \omega L/R \)

- There are two components to this solution, a transient and steady-state
  - The steady-state component is a sinusoid with the same frequency as the source, but has different maximum magnitude and phase angle
Definition of Laplace Transforms

• The Laplace transform of a function is...

\[ L \{ f(t) \} = \int_{0}^{\infty} f(t) e^{-st} dt \]

\[ F(s) = L \{ f(t) \} \]

– Notice, the Laplace transform is a function of \( s \)
– The Laplace transform transforms the problem from the time domain to the frequency domain. This transformation can help us analyze circuits (and mathematical manipulations) more easily.
– It is important to note that the Laplace transform of \( f(t) \) only determines the behavior of \( f(t) \) for \( t>0 \) from its definition.

• Now, let's look at different types of functions and their Laplace transforms.
Step Function

- The step function, illustrated below, is zero for $t < 0$ and has some magnitude $K$ for $t > 0$. A unit step function, denoted as $u(t)$ has a step magnitude of 1.

- In this example, $f(t) = Ku(t)$

\[
\begin{align*}
Ku(t) &= 0, \quad t < 0 \\
Ku(t) &= K, \quad t > 0
\end{align*}
\]

- The step function is not defined at $t = 0$, but in cases we need to define the transition between $0^-$ and $0^+$, we assume that it is linear and that

\[
Ku(0) = 0.5K
\]

- This discontinuous step does not have to occur at $t = 0$. For a step that occurs at $t = a$, we express the step function as $Ku(t - a)$

\[
\begin{align*}
Ku(t-a) &= 0, \quad t < a \\
Ku(t-a) &= K, \quad t > a
\end{align*}
\]
• A step function can also be equal to \( K \) for \( t < a \) and equal to 0 for \( t > a \).
  
  \[
  Ku(a-t) = K, \quad t < a \\
  Ku(a-t) = 0, \quad t > a
  \]

• One can also create arbitrary functions using steps. For instance, a finite-width pulse.

\[
K[u(t-1) - u(t-3)]
\]
Impulse Function

- The impulse function $\delta(t)$ is a signal of infinite magnitude and zero duration
  - A useful mathematical model that enables us to define the derivative at a discontinuity
  - Mathematically,
    \[
    \int_{-\infty}^{\infty} K\delta(t)dt = K
    \]
    \[
    \delta(t) = 0, \; t \neq 0
    \]
  - An impulse at $t = a$ is denoted
    \[
    K\delta(t-a)
    \]
  - Sifting property…
    \[
    \int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a)
    \]
  - The Laplace transform $\delta(t)$ of is…
    \[
    L\{\delta(t)\} = \int_{0^{-}}^{\infty} \delta(t)e^{-st}dt = \int_{0^{-}}^{\infty} \delta(t)dt = 1
    \]
Functional Transforms

• A functional transform is the Laplance transform of a specified function of $t$
  – Note that we are limiting our discussion to unilateral, on-sided transforms which means that all functions are zero for $t < 0$
• The Laplace transform of a unit step function $u(t)$ is…
  \[
  \mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)e^{-st} \, dt = \int_{0^+}^{\infty} 1 \cdot e^{-st} \, dt = \left. \frac{e^{-st}}{-s} \right|_{0^+}^{\infty} = \frac{1}{s}
  \]
• The Laplace transform of a decaying exponential is…
  \[
  \mathcal{L}\{e^{-at}\} = \int_{0^-}^{\infty} e^{-at} \cdot e^{-st} \, dt = \int_{0^+}^{\infty} e^{-(a+s)t} \, dt = \frac{1}{s + a}
  \]
• The Laplace transform of a sinusoid is…
  \[
  \mathcal{L}\{\sin \omega t\} = \int_{0^-}^{\infty} (\sin \omega t) \cdot e^{-st} \, dt = \frac{\omega}{s^2 + \omega^2}
  \]
• A list of useful function transforms can be found in the course web page where there are links to additional transforms.

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Operational Transforms

- Operational transforms show how mathematical functions convert with the Laplace transform. Here is a list of examples...

<table>
<thead>
<tr>
<th></th>
<th>f(t)</th>
<th>F(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>$c_1 f_1(t) + c_2 f_2(t)$</td>
<td>$c_1 F_1(s) + c_2 F_2(s)$</td>
</tr>
<tr>
<td>Differentiation</td>
<td>$\frac{d}{d t} f(t)$</td>
<td>$s F(s) - f(0^-)$</td>
</tr>
<tr>
<td>n-Fold differentiation</td>
<td>$\frac{d^n}{d t^n} f(t)$</td>
<td>$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - s f^{(n-2)}(0^-) - f^{(n-1)}(0^-)$</td>
</tr>
<tr>
<td>Integration</td>
<td>$\int_{0^-}^{t} f(\tau) \ d \tau$</td>
<td>$F(s)/s$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$f(t - t_0) u(t - t_0)$, $t_0 &gt; 0$</td>
<td>$\exp(-st_0) F(s)$</td>
</tr>
<tr>
<td>Frequency shift</td>
<td>$\exp(-s_0 t) f(t)$</td>
<td>$F(s + s_0)$</td>
</tr>
<tr>
<td>Time-frequency scaling</td>
<td>$f(c t)$, $c &gt; 0$</td>
<td>$\frac{1}{c} F\left(\frac{s}{c}\right)$</td>
</tr>
<tr>
<td>t-Multiplication</td>
<td>$t \ f(t)$</td>
<td>$-\frac{d}{d s} F(s)$</td>
</tr>
<tr>
<td>n-Fold t-multiplication</td>
<td>$t^n \ f(t)$</td>
<td>$(-1)^n \frac{d^n}{d s^n} F(s)$</td>
</tr>
</tbody>
</table>
Using the Laplace Transform

- We can use the Laplace transform to solve the RLC circuits we saw earlier.

\[
\frac{v(t)}{R} + \frac{1}{L} \int_0^t v(x)dx + C \frac{dv(t)}{dt} = I_{dc} u(t)
\]

- Now, take the Laplace transform of both sides

\[
\frac{V(s)}{R} + \frac{1}{L} \frac{V(s)}{s} + C [sV(s) - v(0^-)] = I_{dc} \left(\frac{1}{s}\right)
\]

- Solving for \(V(s)\) gives

\[
V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}
\]

- To find \(v(t)\), take the inverse Laplace transform

\[
v(t) = L^{-1}\{V(s)\}
\]
Inverse Laplace Transforms

- $V(s)$ that was just calculated is a **rational** function of $s$, meaning that it can be expressed as a ratio of two polynomials. Thus, it is in the general form:

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}$$

- if $m > n$ then a proper rational function
- if $m \leq n$ then a improper rational function

- The inverse Laplace transform requires finding the partial fraction expansion and then using the operational transforms. For example, take the following rational function:

$$F(s) = \frac{s + a}{s(s + b)(s + c)^2}$$

- this function has four roots and can be rewritten as

$$F(s) = \frac{s + a}{s(s + b)(s + c)^2} = \frac{K_1}{s} + \frac{K_2}{s + b} + \frac{K_3}{s + c} + \frac{K_4}{(s + c)^2}$$

- then,

$$f(t) = L^{-1}\{F(s)\} = \left(K_1 + K_2 e^{-bt} + K_3 e^{-ct} + K_4 t e^{-ct}\right)u(t)$$
Useful Transform Pairs

• Since we will not spend much time solving the inverse Laplace transform, we will stop at this point and present some links to useful transform pairs that you can use (when needed).
  – Table of Laplace transform properties
    [link](http://deas.harvard.edu/courses/es154/2001/Laplace/Table_prop.html)
  – Table of Laplace transform pairs
    [link](http://deas.harvard.edu/courses/es154/2001/Laplace/Table_pairs.html)
  – Table of Laplace transforms of common wave forms
    [link](http://deas.harvard.edu/courses/es154/2001/Laplace/laplace_cwf/laplace_cwf.html)
  – Table of Laplace transforms of basic functions
    [link](http://deas.harvard.edu/courses/es154/2001/Laplace/laplace_basic/laplace_basic.html)
  – Table of Laplace transforms of trigonometric functions
    [link](http://deas.harvard.edu/courses/es154/2001/Laplace/laplace_trig/laplace_trig.html)
Poles and Zeros of F(s)

- Rewriting a rational function as the ratio of two factored polynomials enables us to identify the poles and zeros of $F(s)$. Later, we will see what poles and zeros mean for circuits. For now, here is the general form...

$$F(s) = \frac{N(s)}{D(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_n)}{(s + p_1)(s + p_2)\cdots(s + p_m)}$$

- The roots of the denominator are poles ($x$) and the roots of the numerator are zeros ($o$).
- The poles and zeros can have both real and imaginary components and we can visualize them as points on a complex s-plane.
- We will see more s-plane graphs when we talk about feedback and root locus plots.
Initial- and Final-Value Theorems

- The initial- and final-value theorems can be useful for determining the behavior of \( f(t) \) from \( F(s) \).
  - the initial-value theorem states...
    \[
    \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)
    \]
  - the final-value theorem states...
    \[
    \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
    \]
Laplace Transform for Circuit Analysis

- Two characteristics make Laplace transforms attractive for circuit analysis
  - transforms a set of linear, constant-coefficient, differential equations into a set of linear polynomial equations
  - automatically introduces initial values so that they are an inherent part of the transform process
- The Laplace transform applied to circuits allows us to analyze circuits in the frequency domain (or s-domain)
  - we will see how the s-domain description of circuit input and output allows us to describe them as transfer functions
Circuit Elements in the s-Domain

- Let’s look at basic circuit elements in the s-domain
  - denote time domain voltage and current with lower-case letters \( v \) and \( i \)
  - denote frequency or s-domain voltage and current with capital \( V \) and \( I \)
L (Ldi/dt)

In response to the question raised in class… The equation in the notes is correct for finding the initial condition of the inductor. Let’s work out the math.

\[ L \left\{ \frac{di}{dt} \right\} = \int_{0}^{\infty} L \frac{di}{dt} e^{-st} dt \]

– Now, integrate by parts…

\[ \int u dv = uv - \int v du \]

if \( u = e^{-st} \) and \( dv = L \frac{di}{dt} \) then \( du = -se^{-st} \) and \( v = Li(t) \)

\[ \int u dv = e^{-st} Li(t) \bigg|_{0}^{\infty} - \int Li(t) (-s) e^{-st} dt \]

\[ \int u dv = e^{-st} Li(t) \bigg|_{0}^{\infty} + sL \int i(t) e^{-st} dt \]

\[ L \left\{ \frac{di}{dt} \right\} = 0 - Li(0) + sLI(s) = sLI(s) - Li(0) \]

You can work out the initial condition for a capacitor in the same manner.
+ Example

• Let’s look at the natural response of an RC using Laplace transform techniques.

\[ V_0 = \frac{1}{sC} I + RI \]

\[ I = \frac{CV_0}{RCs+1} = \frac{V_0}{R} \left( \frac{1}{s + (1/RC)} \right) \]

• The inverse transform (by inspection, table look-up) gives

\[ i = \frac{V_0}{R} e^{-t/RC} u(t) \]

\[ v = iR = V_0 e^{-t/RC} u(t) \]
The Transfer Function

- The **transfer function** is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input source.
  - When computing transfer functions, we assume zero initial conditions.
  - We will be using the transfer function extensively to describe circuits throughout this course.

- The transfer function is

\[ H(s) = \frac{Y(s)}{X(s)} \]

  - where \( Y(s) \) is the Laplace transform of the output signal and \( X(s) \) is the Laplace transform of the input signal.
  - Note: The transfer function depends on what is defined as the output and input signals.
  - If there are multiple independent sources, we can use superposition to find response to all of the sources.
+ Example of $H(s)$

- Transfer function examples for a series $RLC$ circuit.

\[
H(s) = \frac{I}{V_g} = \frac{1}{R + sL + \frac{1}{sC}} = \frac{sC}{s^2LC + RCs + 1}
\]

\[
H(s) = \frac{V}{V_g} = \frac{1}{R + sL + \frac{1}{sC}} = \frac{1}{s^2LC + RCs + 1}
\]
Impulse Response

• If a unit impulse drives a circuit, the response of the circuit equals the inverse transform of the transfer function.

\[
\begin{align*}
\text{if } x(t) &= \delta(t), \text{ then } X(s) &= 1 \\
\text{and } Y(s) &= H(s)X(s) = H(s) \\
\text{so, } y(t) &= h(t)
\end{align*}
\]

– The impulse response of a circuit, \(h(t)\), can be used to compute the response of the circuit to any source that drives the circuit.
Next Time

• Supplemental Reading:
  – Nilsson Chapters 13, 14.6-7

• Background
  – We will finish this series of background material on circuit analysis tools by spending a little more time on transfer functions and then seeing how we can graphically view transfer functions and frequency response with Bode plots.